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Mathematical Models and Methods in Applied Sciences

REDUCED BASIS *A POSTERIORI* ERROR BOUNDS FOR THE STOKES EQUATIONS IN PARAMETRIZED DOMAINS: A PENALTY APPROACH

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We present reduced basis approximations and associated rigorous *a posteriori* error bounds for the Stokes equations in parametrized domains. The method, built upon the penalty formulation for saddle-point problems, provides error bounds not only for the velocity but also for the pressure approximation, while simultaneously admitting affine geometric variations with relative ease. The essential ingredients are: *i*) dimension reduction through Galerkin projection onto a low-dimensional reduced basis space; *ii*) stable, good approximation of the pressure through supremizer-enrichment of the velocity reduced basis space; *iii*) optimal and numerically stable approximations identified through an efficient greedy sampling method; *iv*) certainty, through rigorous *a posteriori* bounds for the errors in the reduced basis approximation; and *v*) efficiency, through an offline-online computational strategy. The method is applied to a flow problem in a two-dimensional channel with a (parametrized) rectangular obstacle. Numerical results show that the reduced basis approximation converges rapidly, the effectivities associated with the (inexpensive) rigorous *a posteriori* error bounds remain good even for reasonably small values of the penalty parameter, and that the effects of the penalty parameter are relatively benign.

Keywords: Stokes equations; saddle-point problems; penalty; reduced order model; reduced basis approximation; a posteriori error estimation; error bounds; greedy sampling; offline-online procedure; successive constraints method; real-time computation.

AMS Subject Classification: 76D07, 41A45, 65N15

1. Introduction

The analysis of the Stokes equations for low-Reynolds number fluid flow is often performed as a stepping stone for the more general Navier-Stokes equations, but nevertheless remains relevant in many engineering applications (see Ref. 18), particularly in the field of microfluidics (see, for example, Refs. 6, 9, 49, 50). Quite often, it is crucial to analyze and understand the effects of geometric parameters on the flow, particularly when the objective is to optimize, control, or characterize the system. We thus explore in this paper one way through which to accelerate parameter-space exploration in such many-query and also real-time contexts — reduced order models.

Reduced order modelling for fluid flow problems has received considerable attention because it can often capture the system behavior at significantly less cost than classical discretization techniques such as the finite element method (see, e.g., Refs. 8, 11, 22, 23, 24, 25, 26, 28, 29, 30, 31, 40). Our particular approach is the *certified* reduced basis method, in which the emphasis is on rigorous *a posteriori* error bounds and "optimal" sampling procedures to provide more uniform, rapid convergence over larger parameter domains.

The certified reduced basis method is well-developed for several classes of partial differential equations. However, for incompressible flow problems involving parametrized domains, there is still room for improvement. Earlier work has established reduced basis approximations and associated rigorous *a posteriori* error bounds for *non-parametrized* domains: see Refs. 12, 32, 52 (respectively, Ref. 27) for treatment of the steady (respectively, unsteady) incompressible Navier-Stokes and Boussinesq equations. In these earlier examples, the absence of geometric variations permitted the reduced basis approximation to be built upon divergence-free spaces. In Refs. 12, 43, 45, 47, some work has also been done in extending these techniques to *parametrized* domains; however, in these earlier examples either rigorous error bounds were not treated, or the simple geometric variations considered are applicable to only a very limited set of problems.

Although the reduced basis method in general readily admits certain classes of geometric variations (see Ref. 46), the particular case of saddle point problems presents additional difficulties: special care must be taken in the construction of stable approximation spaces (see Refs. 42, 44, 47), and the error bounds (more specifically, the required lower bounds for the inf-sup stability constants) become increasingly expensive and complicated as the geometric variations become more complex, and especially in nonlinear problems such as the Navier-Stokes equations.

This paper focuses on the development of rapidly convergent reduced basis approximations and associated rigorous *a posteriori* error bounds for the Stokes equations based on a penalty formulation. The penalty method, first introduced in the context of constrained optimization, gained popularity in the 1980s and 1990s as a numerical solution method for some saddle point problems such as the Stokes equations (see, for example, Refs. 3, 10, 20, 34, 35, and the review papers, Refs. 19,

41). Combined with reduced-integration techniques (see Ref. 54), penalty methods "allow" the elimination of the pressure and circumvention of the Babuska-Brezzi inf-sup condition (see Ref. 5). The development here — based upon previous work on reduced basis methods for noncoercive problems in general (see, e.g., Refs. 16, 48, 53) and for the Stokes and Navier-Stokes equations in particular (see Refs. 32, 42, 47, 52) — employs the penalty approximation for a different reason: it not only allows us to approximate the pressure as well as the velocity but in addition enables the development of rigorous error bounds for the reduced basis approximation which do not depend on the (finite element) inf-sup stability constant. The expensive evaluation of (a lower bound to) the inf-sup constant is thus avoided and more general parametric variations can therefore be considered with relative ease. However, these benefits do not come without expense: the penalty approximation introduces an $O(\epsilon)$-error in the velocity and pressure truth approximations, where ϵ denotes the penalty parameter. Nevertheless, for engineering problems of interest these errors are generally acceptable.

Our certified reduced basis approach for the Stokes equations in parametrized domains has several ingredients. First, we provide dimension reduction through Galerkin projection onto a low-dimensional reduced basis space. Second, we achieve stable, good approximation of the pressure through supremizer-enrichment of the velocity reduced basis space. Third, we identify optimal and numerically stable approximations through an efficient greedy sampling method. Fourth, we provide certainty through rigorous *a posteriori* bounds for the errors in the reduced basis approximation. And finally, we achieve efficiency through an offline-online computational strategy. The method is thus suitable in the real-time (e.g., parameter-estimation, control) and many-query (e.g., design optimization, uncertainty quantification) contexts.

The paper is organized as follows. In Section 2 we introduce the model problem, formulate the governing Stokes equations in their variational form, and discuss the penalty formulation. In Section 3 we describe our reduced basis (RB) approximation for the Stokes problem with particular emphasis on approximation stability. In Section 4 we develop rigorous *a posteriori* error bounds. In Section 5 we present numerical results for a flow problem in a two-dimensional channel with a (parametrized) rectangular obstacle, illustrating convergence, computational savings, error bound effectivity, and the relatively benign effects of the penalty parameter. Finally, in Section 6, we provide some concluding remarks.

2. Problem Formulation

2.1. *Strong Formulation on the Parametrized Domain*

We consider Stokes or creeping flow in a two-dimensional channel with an obstacle as illustrated in Fig. 1. We assume that the channel has a fixed aspect ratio, $\mathcal{A} = 4$, and that the obstacle has variable width μ_1 and height μ_2. We further assume that our two-tuple parameter $\mu \equiv (\mu_1, \mu_2)$ is in a prescribed (bounded) parameter domain

$\mathcal{D} \equiv [0.1, 0.5]^2 \subset \mathrm{R}^2$. The physical domain is thus $\tilde{\Omega}(\mu) \equiv]0, \mathcal{A}[\times]0, 1[\setminus \tilde{\mathcal{O}}(\mu)$, where the obstacle is given by $\tilde{\mathcal{O}}(\mu) \equiv](\mathcal{A} - \mu_1)/2, (\mathcal{A} + \mu_1)/2[\times]0, \mu_2[$; the boundary of $\tilde{\Omega}(\mu)$ is denoted by $\tilde{\Gamma}(\mu)$. We assume fully-developed flow conditions with a parabolic velocity profile on the inflow boundary, $\tilde{\Gamma}_{\text{in}}$, and natural outflow conditions on $\tilde{\Gamma}_{\text{out}}$. We further assume no-penetration and no-slip velocity conditions on $\tilde{\Gamma}_0(\mu)$, where $\tilde{\Gamma}_0(\mu) \equiv \tilde{\Gamma}(\mu) \setminus (\tilde{\Gamma}_{\text{in}} \cup \tilde{\Gamma}_{\text{out}})$ consists of the top and bottom boundary of the channel as well as the boundary of the obstacle.

The Stokes equations for the (total) velocity, $\tilde{u}_{\text{t}}(\mu) : \tilde{\Omega}(\mu) \to \mathrm{R}^2$, and the pressure, $\tilde{p}_{\text{e}}(\mu) : \tilde{\Omega}(\mu) \to \mathrm{R}$ at any point $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ in the domain $\tilde{\Omega}(\mu)$ are then given by

$$-\frac{\partial}{\partial \tilde{x}_j}\left(\frac{\partial}{\partial \tilde{x}_j}\tilde{u}_{\text{t}\,i}(\mu)\right) + \frac{\partial}{\partial \tilde{x}_i}\tilde{p}_{\text{e}}(\mu) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial \tilde{x}_i}\tilde{u}_{\text{t}\,i}(\mu) = 0, \tag{2.2}$$

with boundary conditions

$$\tilde{u}_{\text{t}\,i}(\mu)|_{\tilde{\Gamma}_0(\mu)} = 0, \qquad \left(\frac{\partial}{\partial \tilde{n}}\tilde{u}_{\text{t}\,i}(\mu) - \tilde{p}_{\text{e}}(\mu)\,\tilde{n}_i\right)\Big|_{\tilde{\Gamma}_{\text{out}}} = 0, \tag{2.3}$$

for $i, j = 1, 2$, and

$$\tilde{u}_{\text{t}\,1}(\mu)|_{\tilde{\Gamma}_{\text{in}}} = 4\,\tilde{x}_2(1 - \tilde{x}_2), \qquad \tilde{u}_{\text{t}\,2}(\mu)|_{\tilde{\Gamma}_{\text{in}}} = 0. \tag{2.4}$$

Here, repeated indices imply summation, and $\tilde{n}$ is the unit outward normal.

We now decompose the total velocity $\tilde{u}_{\text{t}}(\mu)$ into two parts, i.e., we let

$$\tilde{u}_{\text{t}}(\mu) = \tilde{u}_{\text{L}} + \tilde{u}_{\text{e}}(\mu), \tag{2.5}$$

where the (given) lifting function $\tilde{u}_{\text{L}} : \tilde{\Omega}(\mu) \to \mathrm{R}^2$ is chosen to satisfy the inhomogeneous Dirichlet boundary conditions on $\tilde{\Gamma}_{\text{in}}$, and homogeneous Dirichlet boundary conditions on $\tilde{\Gamma}_0(\mu)$, and $\tilde{u}_{\text{e}}(\mu)$ satisfies homogeneous Dirichlet conditions on

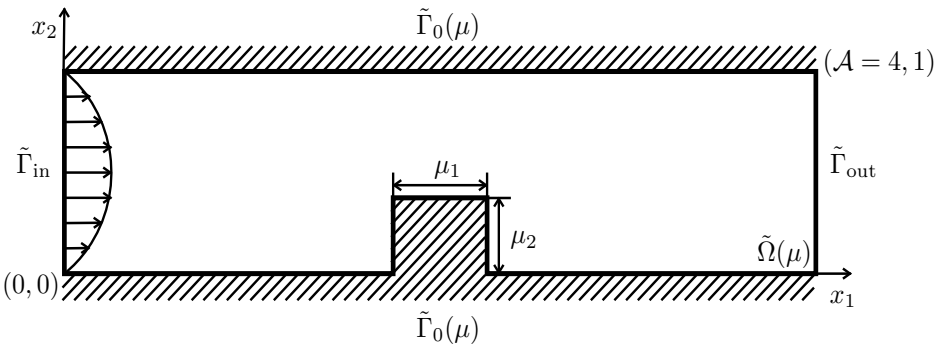

Fig. 1. Computational domain: a parametrized two-dimensional microchannel with an obstacle. The parameters of interest are the width (μ_1) and height (μ_2) of the obstacle.

$\tilde{\Gamma}_0(\mu) \cup \tilde{\Gamma}_{\rm in}$. For our model problem, we choose:

$$\tilde{u}_{\rm L} = \begin{cases} \big(4\,\tilde{x}_2(1-\tilde{x}_2)(1-\tilde{x}_1), 0\big), & \text{on } \tilde{\Omega}_{\rm L} \equiv]0,1[\times]0,1[\,, \\ (0,0), & \text{on } \tilde{\Omega}(\mu)\backslash\tilde{\Omega}_{\rm L}. \end{cases}$$

Note that $\tilde{\Omega}_{\rm L}$ does not include the μ-dependent part of the domain, and therefore $\tilde{u}_{\rm L}$ (on $\tilde{\Omega}_{\rm L}$) is, by construction, parameter-independent.

We now introduce a penalty term to the continuity equation (2.2),

$$\frac{\partial}{\partial \tilde{x}_i}\tilde{u}^\epsilon_{{\rm t}\,i}(\mu) = -\epsilon\,\tilde{p}^\epsilon_{\rm e}(\mu), \tag{2.6}$$

for some small penalty parameter ϵ. We recall that (2.6) essentially approximates the fluid as *nearly* incompressible. From (2.1), (2.3)-(2.6), and our choice of $\tilde{u}_{\rm L}$, our governing equations for $\tilde{u}^\epsilon_{\rm e}(\mu)$ and $\tilde{p}^\epsilon_{\rm e}(\mu)$ are then:

$$-\frac{\partial}{\partial \tilde{x}_j}\left(\frac{\partial}{\partial \tilde{x}_j}\tilde{u}^\epsilon_{{\rm e}\,i}(\mu)\right) + \frac{\partial}{\partial \tilde{x}_i}\tilde{p}^\epsilon_{\rm e}(\mu) = \frac{\partial}{\partial \tilde{x}_j}\left(\frac{\partial}{\partial \tilde{x}_j}\tilde{u}_{{\rm L}\,i}\right), \tag{2.7}$$

$$\frac{\partial}{\partial \tilde{x}_i}\tilde{u}^\epsilon_{{\rm e}\,i}(\mu) + \epsilon\,\tilde{p}^\epsilon_{\rm e}(\mu) = -\frac{\partial}{\partial \tilde{x}_i}\tilde{u}_{{\rm L}\,i}, \tag{2.8}$$

with boundary conditions

$$\tilde{u}^\epsilon_{{\rm e}\,i}(\mu)|_{\tilde{\Gamma}_0(\mu)\cup\tilde{\Gamma}_{\rm in}} = 0, \qquad \left(\frac{\partial}{\partial \tilde{n}}\tilde{u}^\epsilon_{{\rm e}\,i}(\mu) - \tilde{p}^\epsilon_{\rm e}(\mu)\,\tilde{n}_i\right)\Big|_{\tilde{\Gamma}_{\rm out}} = 0, \tag{2.9}$$

for $i, j = 1, 2$; here we have used the fact that on $\tilde{\Gamma}_{\rm out}$, $\partial\tilde{u}_{{\rm L}\,i}(\mu)/\partial\tilde{n} = 0$, $i = 1, 2$. We note that by taking $\epsilon = 0$ in the "penalized" Stokes equations (2.7)-(2.9), we recover the solution $(\tilde{u}_{\rm e}(\mu), \tilde{p}_{\rm e}(\mu))$ to the "classical" Stokes equations (2.1)-(2.3).

2.2. *Weak Formulation on a Reference Domain*

As is standard in reduced basis methods, we now transform our problem statement over the physical domain $\tilde{\Omega}(\mu)$ to an equivalent problem posed over a *parameter-independent* reference domain Ω. For our model problem, we choose $\Omega = \tilde{\Omega}(\mu^{\rm ref})$ where $\mu^{\rm ref} \equiv (\mu_1^{\rm ref}, \mu_2^{\rm ref}) = (0.3, 0.3) \in \mathcal{D}$. The boundary of Ω is then $\Gamma \equiv \Gamma_0 \cup \Gamma_{\rm in} \cup \Gamma_{\rm out}$ where $\Gamma_0 = \tilde{\Gamma}_0(\mu^{\rm ref})$, $\Gamma_{\rm in} = \tilde{\Gamma}_{\rm in}$, $\Gamma_{\rm out} = \tilde{\Gamma}_{\rm out}$. (Further details of the mapping procedure are provided in Appendix A.)

We now introduce several notations required for the remainder of this paper. We define the function spaces

$$X_{\rm e} \equiv \left\{ v \in (H^1(\Omega))^2 \,\middle|\, v|_{\Gamma_0\cup\Gamma_{\rm in}} = 0 \right\}, \tag{2.10}$$

$$Y_{\rm e} \equiv L^2(\Omega); \tag{2.11}$$

here $H^1(\Omega) = \{v \,|\, v \in L^2(\Omega), \nabla v \in (L^2(\Omega))^2\}$, and $L^2(\Omega)$ is the space of square integrable functions over Ω. We also define, for any $w, v \in X_{\rm e}$ and $q, \psi \in Y_{\rm e}$, the

inner products and associated induced norms

$$\begin{aligned}(w,v)_{X_\mathrm{e}} &= \int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}, \qquad \|\cdot\|_{X_\mathrm{e}} = (\cdot,\cdot)_{X_\mathrm{e}}^{1/2},\\ (q,\psi)_{Y_\mathrm{e}} &= \int_\Omega q\,\psi, \qquad\qquad \|\cdot\|_{Y_\mathrm{e}} = (\cdot,\cdot)_{Y_\mathrm{e}}^{1/2}.\end{aligned} \tag{2.12}$$

We can now state the parametrized weak formulation of the governing penalty Stokes equations: for any parameter μ in the parameter domain $\mathcal{D}$, and given a lifting function $u_\mathrm{L} \in H^1(\Omega)$, the (homogeneous part of the) velocity $u_\mathrm{e}^\epsilon(\mu)$ and the corresponding pressure $p_\mathrm{e}^\epsilon(\mu)$ satisfy

$$a(u_\mathrm{e}^\epsilon(\mu), v; \mu) + b(v, p_\mathrm{e}^\epsilon(\mu); \mu) = f(v;\mu), \quad \forall\, v \in X_\mathrm{e}, \tag{2.13}$$

$$b(u_\mathrm{e}^\epsilon(\mu), q; \mu) - \epsilon\, c(p_\mathrm{e}^\epsilon(\mu), q; \mu) = g(q;\mu), \quad \forall\, q \in Y_\mathrm{e}; \tag{2.14}$$

the continuous bilinear forms $a(\cdot,\cdot;\mu)$, $b(\cdot,\cdot;\mu)$, and $c(\cdot,\cdot;\mu)$, and the bounded linear forms $f(\cdot;\mu)$ and $g(\cdot;\mu)$ for our particular model problem are given in Appendix A. We assume that the bilinear forms $a : X_\mathrm{e} \times X_\mathrm{e} \times \mathcal{D} \to \mathrm{R}$ and $c : Y_\mathrm{e} \times Y_\mathrm{e} \times \mathcal{D} \to \mathrm{R}$ are symmetric and coercive,

$$\alpha_a^\mathrm{e}(\mu) \equiv \inf_{v\in X_\mathrm{e}} \frac{a(v,v;\mu)}{\|v\|_{X_\mathrm{e}}^2} > 0, \quad \forall\, \mu \in \mathcal{D}, \tag{2.15}$$

$$\alpha_c^\mathrm{e}(\mu) \equiv \inf_{q\in Y_\mathrm{e}} \frac{c(q,q;\mu)}{\|q\|_{Y_\mathrm{e}}^2} > 0, \quad \forall\, \mu \in \mathcal{D}; \tag{2.16}$$

the bilinear form $b : X_\mathrm{e} \times Y_\mathrm{e} \times \mathcal{D} \to \mathrm{R}$ satisfies the inf-sup condition,

$$\beta^\mathrm{e}(\mu) \equiv \inf_{q\in Y_\mathrm{e}} \sup_{v\in X_\mathrm{e}} \frac{b(v,q;\mu)}{\|q\|_{Y_\mathrm{e}}\|v\|_{X_\mathrm{e}}} > 0, \quad \forall\, \mu \in \mathcal{D}; \tag{2.17}$$

and that a, b, and c are continuous,

$$\gamma_a^\mathrm{e}(\mu) \equiv \sup_{w\in X_\mathrm{e}} \sup_{v\in X_\mathrm{e}} \frac{a(w,v;\mu)}{\|w\|_{X_\mathrm{e}}\|v\|_{X_\mathrm{e}}} < \infty, \quad \forall\, \mu \in \mathcal{D}, \tag{2.18}$$

$$\gamma_b^\mathrm{e}(\mu) \equiv \sup_{q\in Y_\mathrm{e}} \sup_{v\in X_\mathrm{e}} \frac{b(v,q;\mu)}{\|q\|_{Y_\mathrm{e}}\|v\|_{X_\mathrm{e}}} < \infty, \quad \forall\, \mu \in \mathcal{D}, \tag{2.19}$$

$$\gamma_c^\mathrm{e}(\mu) \equiv \sup_{q\in Y_\mathrm{e}} \sup_{r\in Y_\mathrm{e}} \frac{c(q,r;\mu)}{\|q\|_{Y_\mathrm{e}}\|r\|_{Y_\mathrm{e}}} < \infty, \quad \forall\, \mu \in \mathcal{D}. \tag{2.20}$$

We further define the combined function space $Z_\mathrm{e} \equiv X_\mathrm{e} \times Y_\mathrm{e}$; for the associated inner product and induced norm, we consider

$$(W,V)_{Z_\mathrm{e},\mu,\epsilon} = (w,v)_{X_\mathrm{e}} + \epsilon\frac{\alpha_c(\mu)}{\alpha_a(\mu)}(\psi,q)_{Y_\mathrm{e}}, \quad \|\cdot\|_{Z_\mathrm{e},\mu,\epsilon} = (\cdot,\cdot)_{Z_\mathrm{e},\mu,\epsilon}^{1/2}, \tag{2.21}$$

for any $W = (w,\psi) \in Z_\mathrm{e}$ and $V = (v,q) \in Z_\mathrm{e}$.

We also denote by $(u_\mathrm{e}(\mu), p_\mathrm{e}(\mu))$ the solution of the weak form of the standard Stokes equations

$$a(u_\mathrm{e}(\mu), v; \mu) + b(v, p_\mathrm{e}(\mu); \mu) = f(v;\mu), \quad \forall\, v \in X_\mathrm{e}, \tag{2.22}$$

$$b(u_\mathrm{e}(\mu), q; \mu) = g(q;\mu), \quad \forall\, q \in Y_\mathrm{e}. \tag{2.23}$$

We recall that the penalty solution $(u_{\rm e}^\epsilon(\mu), p_{\rm e}^\epsilon(\mu))$ to (2.13)-(2.14) converges to $(u_{\rm e}(\mu), p_{\rm e}(\mu))$ as ϵ goes to zero, and that

$$\|u_{\rm e}(\mu) - u_{\rm e}^\epsilon(\mu)\|_{X_{\rm e}} \le C_{\rm e}(\mu)\,\epsilon, \qquad \|p_{\rm e}(\mu) - p_{\rm e}^\epsilon(\mu)\|_{Y_{\rm e}} \le \frac{C_{\rm e}(\mu)}{\beta^{\rm e}(\mu)}\,\epsilon, \tag{2.24}$$

where the constant $C_{\rm e}(\mu)$ depends only on f, g, $\alpha_a^{\rm e}$, $\gamma_a^{\rm e}$, $\alpha_c^{\rm e}$, $\gamma_c^{\rm e}$, and $\beta^{\rm e}$. We refer the reader to Refs. 3, 34, and 35, and to the standard texts, e.g., Refs. 5, 14, and 39, for more details.

We now assume that the bilinear and linear forms depend affinely on the parameter μ; for instance, we assume that for some finite, small Q_a, we can write $a(\cdot,\cdot;\mu)$ as

$$a(w, v; \mu) = \sum_{k=1}^{Q_a} \Theta_a^k(\mu) a^k(w, v), \quad \forall\, w, v \in X_{\rm e}, \tag{2.25}$$

where the parameter-dependent functions $\Theta_a^k(\mu)$, $1 \le k \le Q_a$, are continuous functions over $\mathcal{D}$, and the parameter-independent bilinear forms $a^k(\cdot,\cdot)$, $1 \le k \le Q_a$, are continuous over $X_{\rm e} \times X_{\rm e}$. We make analogous assumptions on the bilinear forms $b(\cdot,\cdot;\mu)$ and $c(\cdot,\cdot;\mu)$, and on the linear forms $f(\cdot;\mu)$ and $g(\cdot;\mu)$. (Further details are provided in Appendix A.) Although this requirement may be relaxed (see, for example, Refs. 2 and 15), our basic strategy for computational efficiency relies on this assumption of affine dependence.

2.3. *Truth Approximation*

We next introduce a regular triangulation $\mathcal{T}_\Omega$ of Ω. We denote by $Z_\mathcal{N} \equiv X_\mathcal{N} \times Y_\mathcal{N}$ the standard conforming P_2-P_1 (quadratic/linear) velocity-pressure Taylor-Hood finite element approximation subspace over $\mathcal{T}_\Omega$. Here, $\mathcal{N}$ denotes the dimension of $Z_\mathcal{N}$. To simplify our notation for the remainder of this paper, we suppress from here on the subscript $\mathcal{N}$ and denote $X \equiv X_\mathcal{N}$, $Y \equiv Y_\mathcal{N}$, $Z \equiv Z_\mathcal{N}$; one must then simply bear in mind that our truth finite element approximation (which lies in Z), is of dimension $\mathcal{N}$ where $\mathcal{N}$ is typically very large. Furthermore, the truth approximation subspaces shall inherit the inner products and norms: $(\cdot,\cdot)_X \equiv (\cdot,\cdot)_{X_{\rm e}}$, $\|\cdot\|_X \equiv \|\cdot\|_{X_{\rm e}}$; $(\cdot,\cdot)_Y \equiv (\cdot,\cdot)_{Y_{\rm e}}$, $\|\cdot\|_Y \equiv \|\cdot\|_{Y_{\rm e}}$; and $(\cdot,\cdot)_{Z,\mu,\epsilon} \equiv (\cdot,\cdot)_{Z_{\rm e},\mu,\epsilon}$, $\|\cdot\|_{Z,\mu,\epsilon} \equiv \|\cdot\|_{Z_{\rm e},\mu,\epsilon}$.

We now define the truth finite element approximation: Given $\mu \in \mathcal{D}$ and a sufficiently small penalty parameter $\epsilon > 0$, we look for $(u^\epsilon(\mu), p^\epsilon(\mu)) \in Z$ such that

$$a(u^\epsilon(\mu), v; \mu) + b(v, p^\epsilon(\mu); \mu) = f(v; \mu), \quad \forall\, v \in X, \tag{2.26}$$

$$b(u^\epsilon(\mu), q; \mu) - \epsilon\, c(p^\epsilon(\mu), q; \mu) = g(q; \mu), \quad \forall\, q \in Y. \tag{2.27}$$

We shall presume that $\mathcal{N}$ is sufficiently large such that $(u^\epsilon(\mu), p^\epsilon(\mu))$ is effectively indistinguishable from $(u_{\rm e}^\epsilon(\mu), p_{\rm e}^\epsilon(\mu))$. We shall build our RB approximation upon the "truth" discretization (2.26) and (2.27), and we shall measure the error in our RB prediction relative to $(u^\epsilon(\mu), p^\epsilon(\mu))$. As we shall observe, the Online cost of the reduced basis evaluations shall be independent of $\mathcal{N}$ and furthermore, our reduced basis formulation is stable as $\mathcal{N} \to \infty$; we may thus choose $\mathcal{N}$ conservatively.

We then have, for all $\mu \in \mathcal{D}$, our truth finite element coercivity constants,

$$\alpha_a(\mu) \equiv \inf_{v \in X} \frac{a(v,v;\mu)}{\|v\|_X^2} > 0, \tag{2.28}$$

$$\alpha_c(\mu) \equiv \inf_{q \in Y} \frac{c(q,q;\mu)}{\|q\|_Y^2} > 0; \tag{2.29}$$

inf-sup stability constant,

$$\beta(\mu) \equiv \inf_{q \in Y} \sup_{v \in X} \frac{b(v,q;\mu)}{\|q\|_Y \|v\|_X} > 0; \tag{2.30}$$

and continuity constants,

$$\gamma_a(\mu) \equiv \sup_{w \in X} \sup_{v \in X} \frac{a(w,v;\mu)}{\|w\|_X \|v\|_X} < \infty, \tag{2.31}$$

$$\gamma_b(\mu) \equiv \sup_{q \in Y} \sup_{v \in X} \frac{b(v,q;\mu)}{\|q\|_Y \|v\|_X} < \infty, \tag{2.32}$$

$$\gamma_c(\mu) \equiv \sup_{q \in Y} \sup_{r \in Y} \frac{c(q,r;\mu)}{\|q\|_Y \|r\|_Y} < \infty. \tag{2.33}$$

Note that we ensure (2.30) by choosing the stable P_2-P_1 Taylor-Hood pair so that the inf-sup stability constant $\beta(\mu)$ is independent of $\mathcal{N}$ (see, e.g., Refs. 17, 51) .

As before, we shall denote by $(u(\mu), p(\mu))$ the solution to (2.26) and (2.27) with $\epsilon = 0$. We again recall that $(u^\epsilon(\mu), p^\epsilon(\mu))$ converges to $(u(\mu), p(\mu))$ as ϵ goes to zero; moreover,

$$\|u(\mu) - u^\epsilon(\mu)\|_X \le C(\mu)\,\epsilon, \qquad \|p(\mu) - p^\epsilon(\mu)\|_Y \le \frac{C(\mu)}{\beta(\mu)}\,\epsilon. \tag{2.34}$$

where the constant C depends only on f, g, α_a, γ_a, α_c, γ_c, and β (see, e.g., Refs. 5, 14, 39).

3. Reduced Basis Approximation

3.1. *Formulation*

We now turn to the reduced basis (RB) approximation; see Refs. 1, 13, 33, 36, and 37 for early work on RB methods, and Refs. 42, 45, and 47 for later RB work on the Stokes equations. We suppose that we are given a set of hierarchical RB pressure approximation subspaces $Y_N \subset Y$ and associated velocity approximation subspaces $X_N \subset X$. Here, $N \in [1, N_{\max}] \subset \mathrm{N}$ denotes the dimension of Y_N; we denote the dimension of X_N by M. In this paper we consider approximation subspaces X_N such that the dimension of X_N is an integer multiple of N, i.e., $M = \dim(X_N) = \rho N$, $\rho \in \mathrm{N}$. We may thus pursue Galerkin projection with respect to (2.26) and (2.27): Given $\mu \in \mathcal{D}$, we find $u_N^\epsilon(\mu) \in X_N$ and $p_N^\epsilon(\mu) \in Y_N$ such that

$$\begin{aligned} a(u_N^\epsilon(\mu), v; \mu) + b(v, p_N^\epsilon(\mu); \mu) &= f(v;\mu), \quad \forall\, v \in X_N, \\ b(u_N^\epsilon(\mu), q; \mu) - \epsilon\, c(p_N^\epsilon(\mu), q; \mu) &= g(q;\mu), \quad \forall\, q \in Y_N. \end{aligned} \tag{3.1}$$

We reiterate that the reduced basis approximation is defined in terms of a particular truth discretization. The goal of the RB approximation is dimension reduction — $\dim(X_N) + \dim(Y_N) \ll \mathcal{N}$ — and associated (online) computational savings for given (certified) accuracy: the online cost to evaluate the RB approximation is typically several orders of magnitude less than the classical finite element approach (see, e.g., Refs. 38, 46).

We now turn to the construction of our approximation spaces X_N and Y_N. As in the finite element method, care must be taken to ensure that the approximation spaces (X_N, Y_N) form a stable pair, i.e.,

$$\beta_N(\mu) = \inf_{q \in Y_N} \sup_{v \in X_N} \frac{b(v,q;\mu)}{\|q\|_Y \|v\|_X} > 0, \quad \forall\, \mu \in \mathcal{D}. \tag{3.2}$$

We thus begin by assuming that we are given a sample $\mathcal{D}_N \equiv \{\mu^n, 1 \le n \le N\}$. For our present purposes, the $\mathcal{D}_N$ can represent any sequence of nested samples in $\mathcal{D}$ (see Ref. 46). (In actual practice, the samples are optimally chosen by a greedy sampling procedure which will be discussed in Sec. 5.) We then define the associated reduced basis pressure approximation space as

$$Y_N \equiv \text{span}\{p^\epsilon(\mu^n), 1 \le n \le N\}, \quad 1 \le N \le N_{\max}, \tag{3.3}$$

$$= \text{span}\{\phi_n, 1 \le n \le N\} \tag{3.4}$$

where the $\phi_n \in Y$, $1 \le n \le N_{\max}$, are mutually $(\cdot,\cdot)_Y$-orthonormal basis functions.

Before we continue, we first require some additional notations. In particular, we define the supremizer operators $T_\mu : Y \to X$ and $T^k : Y \to X$, $1 \le k \le Q_b$, as

$$(T_\mu q, v)_X = b(v,q;\mu), \qquad (T^k q, v)_X = b^k(v,q), \tag{3.5}$$

for all $v \in X$ and $q \in Y$, where the bilinear forms $b^k(\cdot,\cdot)$, $1 \le k \le Q_b$, are obtained from an affine decomposition of $b(\cdot,\cdot;\mu)$ analogous to (2.25) (see Appendix A for more details). It then follows that

$$T_\mu q = \arg\sup_{v \in X} \frac{b(v,q;\mu)}{\|v\|_X}, \qquad T^k q = \arg\sup_{v \in X} \frac{b^k(v,q)}{\|v\|_X}, \qquad \forall\, q \in Y, \tag{3.6}$$

from the Riesz representation theorem, and that

$$T_\mu q = \sum_{k=1}^{Q_b} \Theta_b^k(\mu) T^k q, \quad \forall q \in Y. \tag{3.7}$$

We now consider two options by which to choose the associated reduced basis velocity approximation space.

3.1.1. *Method 1*

In Method 1 (see Ref. 47), given $\mathcal{D}_N$ (and therefore Y_N), $1 \le N \le N_{\max}$, we choose $X_N = X_N^1$ where

$$X_N^1 \equiv \text{span}\{u^\epsilon(\mu^n),\, T^k p^\epsilon(\mu^n),\, 1 \le n \le N,\, 1 \le k \le Q_b\}$$

$$= \text{span}\left\{\xi_m^1, 1 \le m \le M\right\}; \tag{3.8}$$

here the $\xi_m \in X, 1 \le m \le M$, are mutually $(\cdot,\cdot)_X$-orthonormal basis functions, and $M = (1+Q_b)N$.

As alluded to in Ref. 47, one can then show that $\beta_N(\mu) \ge \beta(\mu) > 0$, for all $\mu \in \mathcal{D}$. We note from (3.7) and (3.8) that $T_\mu q \in X_N$ for all $q \in Y_N$, and therefore

$$\begin{aligned} 0 < \beta(\mu) &= \inf_{q\in Y} \sup_{v\in X} \frac{b(v,q;\mu)}{\|q\|_Y \|v\|_X} \\ &\le \inf_{q\in Y_N} \sup_{v\in X} \frac{b(v,q;\mu)}{\|q\|_Y \|v\|_X} \\ &= \inf_{q\in Y_N} \frac{b(T_\mu q,q;\mu)}{\|q\|_Y \|T_\mu q\|_X} \\ &= \inf_{q\in Y_N} \sup_{v\in X_N} \frac{b(v,q;\mu)}{\|q\|_Y \|v\|_X} \\ &= \beta_N(\mu), \end{aligned}$$

for all $\mu \in \mathcal{D}$. It then follows that the reduced basis approximation spaces (X_N^1, Y_N) constitute an inf-sup stable pair.

3.1.2. *Method 2*

In Method 2 (again see Ref. 47), given $\mathcal{D}_N$ (and therefore Y_N), $1 \le N \le N_{\max}$, we choose $X_N = X_N^2$, where

$$\begin{aligned} X_N^2 &= \text{span}\left\{u^\epsilon(\mu^n),\, T_{\mu^n} p^\epsilon(\mu^n),\, 1 \le n \le N\right\} \\ &\equiv \text{span}\left\{\xi_m^2, 1 \le m \le M\right\}; \end{aligned} \tag{3.9}$$

here the $\xi_m^2 \in X, 1 \le m \le M$, are again mutually $(\cdot,\cdot)_X$-orthonormal basis functions, and $M = 2N$.

For Method 2, the dimension of the reduced basis (velocity) space is significantly smaller; however, one can no longer demonstrate a stability result as for Method 1, and we cannot claim stability of (X_N^2, Y_N) *a priori*. Nevertheless, even for this choice of reduced basis approximation spaces, problem (3.1) remains well-posed due to the regularizing effect of the penalty term (see, e.g., Ref. 5 for more details).

3.2. *Offline-Online Computational Procedure*

Since much of the required machinery is by now standard in reduced basis methods (see, for example, Ref. 46 and references therein), we only briefly summarize the Offline-Online procedure and associated computational costs.

The basic strategy relies on the affine dependence of the operators (see (2.25) and Appendix A). Noting that for any $\mu \in \mathcal{D}$, we can expand $u_N^\epsilon(\mu)$ and $p_N^\epsilon(\mu)$ as

$$u_N^\epsilon(\mu) = \sum_{m=1}^{M} u_{N\,m}^\epsilon(\mu)\xi_m, \qquad p_N^\epsilon(\mu) = \sum_{n=1}^{N} p_{N\,n}^\epsilon(\mu)\phi_n, \tag{3.10}$$

the reduced basis equations (3.1) may thus be written as

$$\sum_{m=1}^{M}\left(\sum_{k=1}^{Q_a}\Theta_a^k(\mu)A_{N\,im}^k\right)u_{N\,m}^{\epsilon}(\mu)+\sum_{n=1}^{N}\left(\sum_{k=1}^{Q_b}\Theta_b^k(\mu)B_{N\,ni}^k\right)p_{N\,n}^{\epsilon}(\mu)$$
$$=\sum_{k=1}^{Q_f}\Theta_f^k(\mu)F_{N\,i}^k,\qquad\text{for all } 1\le i\le M,$$

$$\sum_{m=1}^{M}\left(\sum_{k=1}^{Q_b}\Theta_b^k(\mu)B_{N\,jm}^k\right)u_{N\,m}^{\epsilon}(\mu)-\epsilon\sum_{n=1}^{N}\left(\sum_{k=1}^{Q_c}\Theta_c^k(\mu)C_{N\,jn}^k\right)p_{N\,n}^{\epsilon}(\mu)$$
$$=\sum_{k=1}^{Q_g}\Theta_g^k(\mu)G_{N\,j}^k,\qquad\text{for all } 1\le j\le N.$$

For $1\le i,m\le M$, and $1\le j,n\le N$, the μ-independent quantities are given by

$$\begin{aligned}A_{N\,im}^k&=a^k(\xi_m,\xi_i),&&1\le k\le Q_a,\\ B_{N\,nm}^k&=b^k(\xi_m,\phi_n),&&1\le k\le Q_b,\\ C_{N\,jn}^k&=c^k(\phi_n,\phi_j),&&1\le k\le Q_c,\\ F_{N\,i}^k&=f^k(\xi_i),&&1\le k\le Q_f,\\ G_{N\,j}^k&=g^k(\phi_j),&&1\le k\le Q_g.\end{aligned}\tag{3.11}$$

Offline, we first form and store the parameter-independent quantities at $O(\mathcal{N}^*)$ computational cost and storage which depends, of course, on $\mathcal{N}$. Online, for any parameter value $\mu\in\mathcal{D}$, we perform the required sums and solve the resulting $(M+N)\times(M+N)$ system of linear equations for $u_N^{\epsilon}(\mu), p_N^{\epsilon}(\mu)$ at $O((N+M)^3)$ computational cost.

The expensive ($\mathcal{N}$-dependent) offline stage, performed once, thus enables the subsequent very inexpensive ($\mathcal{N}$-independent) online stage. The reduced basis approach is therefore particularly relevant in the real-time, many-query contexts of optimization, control, and inverse problems.

4. *A Posteriori* Error Estimation

We aim to develop not only efficient reduced order approximations, but also rigorous, sharp and inexpensive *a posteriori* error bounds. While the required theoretical ingredients already exist for noncoercive problems in general (see, for example, Refs. 16, 53) and flow problems in particular (see, for example, Refs. 12, 16, 32, 47, 52), the required lower bound to the inf-sup stability constant (see Ref. 21) poses some difficulties, particularly when one considers the nonlinear Navier-Stokes equations.

Using the penalty approximation as our point of departure, we thus develop rigorous *a posteriori* error bounds which *do not* require lower bounds to a global inf-sup stability constant.

4.1. *Formulation*

We let $e_N^{u,\epsilon}(\mu) \in X$ and $e_N^{p,\epsilon}(\mu) \in Y$ be the errors in our reduced basis approximation,

$$e_N^{u,\epsilon}(\mu) \equiv u^\epsilon(\mu) - u_N^\epsilon(\mu), \tag{4.1}$$
$$e_N^{p,\epsilon}(\mu) \equiv p^\epsilon(\mu) - p_N^\epsilon(\mu); \tag{4.2}$$

furthermore, we denote by $\mathrm{E}_N^\epsilon(\mu)$ the norm of the combined error in our reduced basis approximation, i.e.,

$$\mathrm{E}_N^\epsilon(\mu) = \|\big(e_N^{u,\epsilon}(\mu), e_N^{p,\epsilon}(\mu)\big)\|_{Z,\mu,\epsilon} = \left(\|e_N^{u,\epsilon}(\mu)\|_X^2 + \epsilon\frac{\alpha_c(\mu)}{\alpha_a(\mu)}\|e_N^{p,\epsilon}(\mu)\|_Y^2\right)^{1/2}. \tag{4.3}$$

We aim to develop an *a posteriori* error bound $\Delta_N^\epsilon(\mu)$ for the error such that

$$\mathrm{E}_N^\epsilon(\mu) \le \Delta_N^\epsilon(\mu), \quad \forall\ \mu \in \mathcal{D}. \tag{4.4}$$

We also introduce the effectivity associated with this error estimator as

$$\eta_N^\epsilon(\mu) = \frac{\Delta_N^\epsilon(\mu)}{\mathrm{E}_N^\epsilon(\mu)}. \tag{4.5}$$

The effectivity is therefore a measure of the quality of the proposed error estimator: effectivities ≥ 1 indicate that the error estimate is *rigorous*; and effectivities close to unity indicate that the error estimate is *sharp*.

To construct our *a posteriori* RB error bound, we need two sets of ingredients. The first set of ingredients consists of lower (and upper bounds) for our truth coercivity constants:

$$\begin{aligned} \alpha_a^{\mathrm{LB}}(\mu) \le \alpha_a(\mu) \le \alpha_a^{\mathrm{UB}}(\mu), \\ \alpha_c^{\mathrm{LB}}(\mu) \le \alpha_c(\mu) \le \alpha_c^{\mathrm{UB}}(\mu), \end{aligned} \qquad \forall\, \mu \in \mathcal{D}, \tag{4.6}$$

where $\alpha_a(\mu)$ and $\alpha_c(\mu)$ are defined in (2.28) and (2.29), respectively.

The second set of ingredients consists of the dual norms of the residuals,

$$\varepsilon_N^{u,\epsilon}(\mu) \equiv \|r_N^{u,\epsilon}(\cdot;\mu)\|_{X'} = \sup_{v\in X} \frac{r_N^{u,\epsilon}(v;\mu)}{\|v\|_X}, \tag{4.7}$$
$$\varepsilon_N^{p,\epsilon}(\mu) \equiv \|r_N^{p,\epsilon}(\cdot;\mu)\|_{Y'} = \sup_{q\in Y} \frac{r_N^{p,\epsilon}(q;\mu)}{\|q\|_Y}, \tag{4.8}$$

where $r_N^{u,\epsilon}(\cdot;\mu) \in X'$ and $r_N^{p,\epsilon}(\cdot;\mu) \in Y'$ are the residuals associated with the reduced basis approximation (3.1), i.e.,

$$r_N^{u,\epsilon}(v;\mu) = f(v;\mu) - a(u_N^\epsilon(\mu), v;\mu) - b(v, p_N^\epsilon(\mu);\mu), \qquad \forall\ v \in X, \tag{4.9}$$
$$r_N^{p,\epsilon}(q;\mu) = g(q;\mu) - b(u_N^\epsilon(\mu), q;\mu) + \epsilon\, c(p_N^\epsilon(\mu), q;\mu), \qquad \forall\ q \in Y. \tag{4.10}$$

Furthermore, from standard duality arguments we have

$$\varepsilon_N^{u,\epsilon}(\mu) = \|\hat{e}_N^{u,\epsilon}(\mu)\|_X, \qquad \varepsilon_N^{p,\epsilon}(\mu) = \|\hat{e}_N^{p,\epsilon}(\mu)\|_Y, \tag{4.11}$$

where $\hat{e}_N^{u,\epsilon}(\mu)$ and $\hat{e}_N^{p,\epsilon}(\mu)$ are the Riesz representations of the linear functionals $r_N^{u,\epsilon}(\cdot;\mu)$ and $r_N^{p,\epsilon}(\cdot;\mu)$, respectively, with

$$(\hat{e}_N^{u,\epsilon}(\mu), v)_X = r_N^{u,\epsilon}(v;\mu), \quad \forall\, v \in X, \tag{4.12}$$

$$(\hat{e}_N^{p,\epsilon}(\mu), q)_Y = r_N^{p,\epsilon}(q;\mu), \quad \forall\, q \in Y. \tag{4.13}$$

We can now state:

Proposition 4.1. *For given $\mu \in \mathcal{D}$, the* a posteriori *error bound*

$$\Delta_N^\epsilon(\mu) \equiv \frac{1}{\alpha_a^{\mathrm{LB}}(\mu)} \left(\varepsilon_N^{u,\epsilon}(\mu)^2 + \frac{\alpha_a^{\mathrm{LB}}(\mu)}{\epsilon\, \alpha_c^{\mathrm{LB}}(\mu)}\, \varepsilon_N^{p,\epsilon}(\mu)^2 \right)^{\frac{1}{2}} \tag{4.14}$$

satisfies (4.4) *for any $N \in [1, N_{\max}]$.*

Proof. (For clarity of exposition, we suppress all μ-dependence in this proof.)

We begin by noting from (3.1) that

$$a(e_N^{u,\epsilon}, v) + b(v, e_N^{p,\epsilon}) = r_N^{u,\epsilon}(v), \quad \forall\, v \in X, \tag{4.15}$$

$$b(e_N^{u,\epsilon}, q) - \epsilon\, c(e_N^{p,\epsilon}, q) = r_N^{p,\epsilon}(q), \quad \forall\, q \in Y. \tag{4.16}$$

Choosing $v = e_N^{u,\epsilon}$ and $q = e_N^{p,\epsilon}$, we obtain

$$a(e_N^{u,\epsilon}, e_N^{u,\epsilon}) + \epsilon\, c(e_N^{p,\epsilon}, e_N^{p,\epsilon}) = r_N^{u,\epsilon}(e_N^{u,\epsilon}) - r_N^{p,\epsilon}(e_N^{p,\epsilon}) \tag{4.17}$$

from which it follows that

$$\begin{aligned} \alpha_a \|e_N^{u,\epsilon}\|_X^2 + \epsilon\, \alpha_c \|e_N^{p,\epsilon}\|_Y^2 &\le \|r_N^{u,\epsilon}\|_{X'} \|e_N^{u,\epsilon}\|_X + \|r_N^{p,\epsilon}\|_{Y'} \|e_N^{p,\epsilon}\|_Y \\ &\le \left[\frac{1}{\alpha_a} \|r_N^{u,\epsilon}\|_{X'}^2 + \frac{1}{\epsilon\, \alpha_c} \|r_N^{p,\epsilon}\|_{Y'}^2 \right]^{\frac{1}{2}} \left[\alpha_a \|e_N^{u,\epsilon}\|_X^2 + \epsilon\, \alpha_c \|e_N^{p,\epsilon}\|_Y^2 \right]^{\frac{1}{2}} \end{aligned}$$

from (2.28), (2.29), (4.7), (4.8), and the Cauchy-Schwarz inequality. The result then directly follows from (4.6), (4.7) and (4.8). □

We now state a corollary which deals with the quality of our error estimator.

Corollary 4.1. *For given $\mu \in \mathcal{D}$ and $\epsilon \in (0, 1]$, the effectivity of the* a posteriori *error bound of* (4.14) *satisfies*

$$1 \le \eta_N^\epsilon(\mu) \le \frac{C_\eta(\mu)}{\sqrt{\epsilon}} \tag{4.18}$$

for any $N \in [1, N_{\max}]$.

Proof. (We again suppress all μ-dependence.)

The first inequality in (4.18) immediately follows from Proposition 4.1. In order to derive the upper bound for the effectivity, we note from (4.15) and (4.16) that the dual norms of the residuals satisfy

$$\begin{aligned} \|r_N^{u,\epsilon}\|_{X'} &\le \gamma_a \|e_N^{u,\epsilon}\|_X + \gamma_b \|e_N^{p,\epsilon}\|_Y \\ &\le \left(\gamma_a^2 + \frac{\alpha_a}{\epsilon\, \alpha_c} \gamma_b^2 \right)^{\frac{1}{2}} \left(\|e_N^{u,\epsilon}\|_X^2 + \frac{\epsilon\, \alpha_c}{\alpha_a} \|e_N^{p,\epsilon}\|_Y^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\|r_N^{p,\epsilon}\|_{Y'} &\leq \gamma_b \|e_N^{u,\epsilon}\|_X + \epsilon\,\gamma_c \|e_N^{p,\epsilon}\|_Y \\ &\leq \left(\gamma_b^2 + \frac{\epsilon\,\alpha_a}{\alpha_c}\gamma_c^2\right)^{\frac{1}{2}} \left(\|e_N^{u,\epsilon}\|_X^2 + \frac{\epsilon\,\alpha_c}{\alpha_a}\|e_N^{p,\epsilon}\|_Y^2\right)^{\frac{1}{2}} \qquad (4.20)\end{aligned}$$

from (2.31), (2.32), (2.33), and the Cauchy-Schwarz inequality. From (4.14), (4.19), and (4.20) we therefore obtain

$$\begin{aligned}\left(\alpha_a^{\mathrm{LB}}\,\Delta_N^\epsilon\right)^2 &= \|r_N^{u,\epsilon}\|_{X'}^2 + \frac{\alpha_a^{\mathrm{LB}}}{\epsilon\,\alpha_c^{\mathrm{LB}}}\,\|r_N^{p,\epsilon}\|_{Y'}^2 \\ &\leq \left(\|e_N^{u,\epsilon}\|_X^2 + \frac{\epsilon\,\alpha_c}{\alpha_a}\|e_N^{p,\epsilon}\|_Y^2\right)\left(\gamma_a^2 + \frac{\alpha_a}{\epsilon\,\alpha_c}\gamma_b^2 + \frac{\alpha_a^{\mathrm{LB}}}{\epsilon\,\alpha_c^{\mathrm{LB}}}\left(\gamma_b^2 + \frac{\epsilon\,\alpha_a}{\alpha_c}\gamma_c^2\right)\right) \\ &= \left(\mathrm{E}_N^\epsilon\right)^2 \left[\frac{1}{\epsilon}\left(\frac{\alpha_a}{\alpha_c} + \frac{\alpha_a^{\mathrm{LB}}}{\alpha_c^{\mathrm{LB}}}\right)\gamma_b^2 + \frac{\alpha_a\,\alpha_a^{\mathrm{LB}}}{\alpha_c\,\alpha_c^{\mathrm{LB}}}\gamma_c^2 + \gamma_a^2\right].\end{aligned}$$

Thus, for $\epsilon \in (0,1]$, it follows that (4.18) holds for

$$C_\eta = \frac{1}{\alpha_a^{\mathrm{LB}}}\left[\left(\frac{\alpha_a}{\alpha_c} + \frac{\alpha_a^{\mathrm{LB}}}{\alpha_c^{\mathrm{LB}}}\right)\gamma_b^2 + \frac{\alpha_a\,\alpha_a^{\mathrm{LB}}}{\alpha_c\,\alpha_c^{\mathrm{LB}}}\gamma_c^2 + \gamma_a^2\right]^{\frac{1}{2}}. \qquad \square$$

4.2. *Offline-Online Computational Procedure*

It is clear from (4.14) that there are two components to the calculation of $\Delta_N^\epsilon(\mu)$: computation of the dual norms of the residuals, $\varepsilon_N^{u,\epsilon}(\mu)$ and $\varepsilon_N^{p,\epsilon}(\mu)$, and evaluation of $\alpha_a^{\mathrm{LB}}(\mu)$ and $\alpha_c^{\mathrm{LB}}(\mu)$. The former is an application of (now) standard reduced basis techniques; we thus provide here only a brief summary. The latter is computed using the Successive Constraints Method which is described in detail in Refs. 21 and 46; we present here a version with a very minimally modified stopping criterion for the greedy algorithm.

4.2.1. *Dual Norm of the Residual*

We first exploit the reduced basis representation (3.10) and the affine dependence (see (2.25) and Appendix A) to express the residuals $r_N^{u,\epsilon}(\cdot;\mu)$ and $r_N^{p,\epsilon}(\cdot;\mu)$ defined in (4.9) and (4.10), respectively, as

$$\begin{aligned}r_N^{u,\epsilon}(v;\mu) = \sum_{k=1}^{Q_f}\Theta_f^k(\mu) f^k(v) &- \sum_{m=1}^{M}\sum_{k=1}^{Q_a} u_{N\,m}^\epsilon(\mu)\,\Theta_a^k(\mu)\,a^k(\xi_m, v) \\ &- \sum_{n=1}^{N}\sum_{k=1}^{Q_b} p_{N\,n}^\epsilon(\mu)\,\Theta_b^k(\mu)\,b^k(v,\phi_n), \quad \forall\, v \in X, \qquad (4.21)\end{aligned}$$

$$r_N^{p,\epsilon}(q;\mu) = \sum_{k=1}^{Q_g} \Theta_g^k(\mu) g^k(q) - \sum_{m=1}^{M}\sum_{k=1}^{Q_b} u_{N\,m}^{\epsilon}(\mu)\,\Theta_b^k(\mu)\,b^k(\xi_m,q) + \epsilon \sum_{n=1}^{N}\sum_{k=1}^{Q_c} p_{N\,n}^{\epsilon}(\mu)\,\Theta_c^k(\mu)\,c^k(\phi_n,q), \quad \forall\, q \in Y. \tag{4.22}$$

This can be written succinctly as

$$r_N^{u,\epsilon}(v;\mu) = \sum_{i=1}^{\mathcal{Q}^u} \Phi_{N\,i}^{u,\epsilon}(\mu)\,\mathcal{F}_{N\,i}^u(v), \quad \forall\, v \in X, \tag{4.23}$$

$$r_N^{p,\epsilon}(q;\mu) = \sum_{i=1}^{\mathcal{Q}^p} \Phi_{N\,i}^{p,\epsilon}(\mu)\,\mathcal{F}_{N\,i}^p(q), \quad \forall\, q \in Y, \tag{4.24}$$

where $\mathcal{Q}^u = Q_f + Q_a M + Q_b N$ and $\mathcal{Q}^p = Q_g + Q_b M + Q_c N$. Here, the $\Phi_{N\,i}^{u,\epsilon}(\mu)$ and $\Phi_{N\,i}^{p,\epsilon}(\mu)$ depend on μ explicitly through the coefficients $\Theta_a(\mu)$, $\Theta_b(\mu)$, $\Theta_c(\mu)$, $\Theta_f(\mu)$, and $\Theta_g(\mu)$, but also implicitly through $u_N^{\epsilon}(\mu)$, and $p_N^{\epsilon}(\mu)$; $\mathcal{F}_{N\,i}^u(v)$ and $\mathcal{F}_{N\,i}^p(q)$ depend only on the ξ_m, $1 \le m \le M$, and ϕ_n, $1 \le n \le N$.

It now follows directly from (4.12) and (4.13) that

$$\hat{e}_N^{u,\epsilon}(\mu) = \sum_{i=1}^{\mathcal{Q}^u} \Phi_{N\,i}^{u,\epsilon}(\mu)\,\Gamma_{N\,i}^u, \tag{4.25}$$

$$\hat{e}_N^{p,\epsilon}(\mu) = \sum_{i=1}^{\mathcal{Q}^p} \Phi_{N\,i}^{p,\epsilon}(\mu)\,\Gamma_{N\,i}^p, \tag{4.26}$$

where the $\Gamma_{N\,i}^u$ and $\Gamma_{N\,i}^p$ are solutions to the *μ-independent* problems

$$(\Gamma_{N\,i}^u, v)_X = \mathcal{F}_{N\,i}^u(v), \quad \forall\, v \in X, \quad 1 \le i \le \mathcal{Q}^u, \tag{4.27}$$

$$(\Gamma_{N\,i}^p, q)_Y = \mathcal{F}_{N\,i}^p(q), \quad \forall\, q \in Y, \quad 1 \le i \le \mathcal{Q}^p. \tag{4.28}$$

It then follows that

$$\left(\varepsilon_N^{u,\epsilon}(\mu)\right)^2 = \left(\hat{e}_N^{u,\epsilon}(\mu), \hat{e}_N^{u,\epsilon}(\mu)\right)_X = \sum_{i=1}^{\mathcal{Q}^u}\sum_{j=1}^{\mathcal{Q}^u} \Phi_{N\,i}^{u,\epsilon}(\mu)\,\Phi_{N\,j}^{u,\epsilon}(\mu)\left(\Gamma_{N\,i}^u, \Gamma_{N\,j}^u\right)_X, \tag{4.29}$$

$$\left(\varepsilon_N^{p,\epsilon}(\mu)\right)^2 = \left(\hat{e}_N^{p,\epsilon}(\mu), \hat{e}_N^{p,\epsilon}(\mu)\right)_Y = \sum_{k=1}^{\mathcal{Q}^p}\sum_{l=1}^{\mathcal{Q}^p} \Phi_{N\,k}^{p,\epsilon}(\mu)\,\Phi_{N\,l}^{p,\epsilon}(\mu)\left(\Gamma_{N\,k}^p, \Gamma_{N\,l}^p\right)_Y. \tag{4.30}$$

We may now summarize the offline-online decomposition.

In the offline stage, we find the $\Gamma_{N\,i}^u$, $1 \le i \le \mathcal{Q}^u$, and $\Gamma_{N\,k}^p$, $1 \le k \le \mathcal{Q}^p$, and form the inner products $(\Gamma_{N\,i}^u, \Gamma_{N\,j}^u)_X$, $1 \le i,j \le \mathcal{Q}^u$, and $(\Gamma_{N\,k}^p, \Gamma_{N\,l}^p)_Y$, $1 \le k,l \le \mathcal{Q}^p$. The operation count for the offline stage clearly depends on $\mathcal{N}$: we must solve $\mathcal{Q}^u$

"velocity-" and $\mathcal{Q}^p$ "pressure-" finite element problems (4.27) and (4.28) and we have to compute $(\mathcal{Q}^u)^2$ (respectively, $(\mathcal{Q}^p)^2$) inner products in X (respectively, in Y).

In the online stage, given the reduced basis coefficients $u^{\epsilon}_{N\,m}(\mu)$, $1 \le m \le M$, and $p^{\epsilon}_{N\,n}(\mu)$, $1 \le n \le N$, we can then readily compute the coefficient functions $\Phi^{u,\epsilon}_{N\,i}(\mu)$, $1 \le i \le \mathcal{Q}^u$, and $\Phi^{p,\epsilon}_{N\,k}(\mu)$, $1 \le k \le \mathcal{Q}^p$. From the stored inner products we then perform the sum (4.29) at $O((\mathcal{Q}^u)^2) = O((Q_f + Q_a M + Q_b N)^2)$ operations, and the sum (4.30) at $O((\mathcal{Q}^p)^2) = O((Q_g + Q_b M + Q_c N)^2)$ operations. As desired, the operation count for the online stage is independent of $\mathcal{N}$.

We now turn to the second component required for the calculation of our error bounds.

4.2.2. *Lower and Upper Bounds for the Coercivity Constants*

We compute the required coercivity lower bounds $\alpha_a^{\mathrm{LB}}(\mu)$ and $\alpha_c^{\mathrm{LB}}(\mu)$ using the successive constraints method (SCM) proposed by Huynh, et al in Ref. 21, with a minimally modified stopping criterion. We shall only state the main result, and refer the reader to Ref. 21 for more details. We also only describe the procedure for $\alpha_a^{\mathrm{LB}}(\mu)$; the method to obtain $\alpha_c^{\mathrm{LB}}(\mu)$ is analogous.

Following Ref. 21, we define

$$\mathcal{Y} \equiv \left\{ y = (y_1, \ldots, y_{Q_a}) \in \mathrm{R}^{Q_a} \,\middle|\, \exists\, v_y \in X \text{ s.t. } y_q = \frac{a^q(v_y, v_y)}{\|v_y\|_X^2}, 1 \le q \le Q_a \right\} \quad (4.31)$$

so that $\alpha_a(\mu)$ may be written as

$$\alpha_a(\mu) = \min_{y \in \mathcal{Y}} \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\, y^q. \quad (4.32)$$

We then define

$$B_q^- \equiv \inf_{v \in X} \frac{a^q(v,v)}{\|v\|_X^2}, \qquad B_q^+ \equiv \sup_{v \in X} \frac{a^q(v,v)}{\|v\|_X^2}, \qquad 1 \le q \le Q_a, \quad (4.33)$$

and $\mathcal{B} \equiv \prod_{q=1}^{Q_a} [B_q^-, B_q^+] \subset \mathrm{R}^{Q_a}$; we also define our training set $\Xi_J^{\mathrm{SCM}} \equiv \{\nu_j \in \mathcal{D},\, j = 1, \ldots, J\}$ and SCM sample $C_K \equiv \{\omega_k \in \mathcal{D},\, k = 1, \ldots, K\}$. Then, for given $L_\alpha \in \mathrm{N}$, we set $L = \min(K, L_\alpha)$ and let $P_L(\mu; C_K) \subseteq C_K$ be the set of L points in C_K closest to μ in the Euclidean norm.

We may now define

$$\mathcal{Y}_{\mathrm{UB}}(C_K) \equiv \left\{ y^*(\omega_k),\, k = 1, \ldots, K \right\}, \quad (4.34)$$

where $y^*(\mu) \equiv \arg\min_{y \in \mathcal{Y}} \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\, y^q$, and

$$\mathcal{Y}_{\mathrm{LB}}(\mu; C_K) \equiv \left\{ y \in \mathcal{B} \,\middle|\, \sum_{q=1}^{Q_a} \Theta_a^q(\mu')\, y^q \ge \alpha_a(\mu'),\ \forall\, \mu' \in P_L(\mu; C_K) \right\} \quad (4.35)$$

so that our coercivity upper and lower bounds are given by

$$\alpha_a^{\rm UB}(\mu;C_K) \equiv \min_{y\in\mathcal{Y}_{\rm UB}(C_K)} \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\, y^q, \qquad \alpha_a^{\rm LB}(\mu;C_K) \equiv \min_{y\in\mathcal{Y}_{\rm LB}(\mu;C_K)} \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\, y^q. \tag{4.36}$$

It now remains to determine C_K by an offline greedy algorithm. We summarize the procedure in Algorithm 1. Note that the greedy criterion

$$\frac{\alpha_a^{\rm UB}(\nu_j;C_K) - \alpha_a^{\rm LB}(\nu_j;C_K)}{|\alpha_a^{\rm LB}(\nu_j;C_K)|} < \delta_{\rm tol}^{\rm SCM} < 1, \tag{4.37}$$

for all $\nu_j \in \Xi_J^{\rm SCM}$, is a slight modification of that in Ref. 21. The criterion (4.37) ensures that for all ν_j in the test sample, $\Xi_J^{\rm SCM}$, (i) $\delta_{\rm tol}^{\rm SCM}$ is an upper bound to the real relative error, $(\alpha_a(\nu_j) - \alpha_a^{\rm LB}(\nu_j;C_K))/\alpha_a(\nu_j)$; and (ii) the lower bound, $\alpha_a^{\rm LB}(\nu_j;C_K)$, is positive.

Algorithm 1 SCM Greedy Algorithm

Choose $\omega_1 \in \mathcal{D}$, $L_\alpha \in \mathrm{N}$, $\Xi_J^{\rm SCM} \subseteq \mathcal{D}$, $\delta_{\rm tol}^{\rm SCM} \in (0,1)$
Compute $\mathcal{B}$
Set $K \leftarrow 0$, $C_K \leftarrow \{\}$
repeat
  $K \leftarrow K+1$
  $C_K \leftarrow C_{K-1} \cup \{\omega_K\}$
  Compute and store $\alpha_a(\omega_K)$, $y^*(\omega_K)$
  for all $\mu \in \Xi_J^{\rm SCM}$ **do**
    Compute $\alpha_a^{\rm LB}(\mu;C_K)$ and $\alpha_a^{\rm UB}(\mu;C_K)$
    $\delta_K(\mu) = \dfrac{\alpha_a^{\rm UB}(\mu;C_K) - \alpha_a^{\rm LB}(\mu;C_K)}{|\alpha_a^{\rm LB}(\mu;C_K)|}$
  end for
  $\omega_{K+1} = \arg\max_{\mu\in\Xi_J^{\rm SCM}} \delta_K(\mu)$
until $\delta_K(\omega_{K+1}) < \delta_{\rm tol}^{\rm SCM}$

We conclude this section by summarizing the cost of the offline-online SCM stages for the computation of $\alpha_a^{\rm LB}(\mu)$; the procedure and cost for $\alpha_c^{\rm LB}(\mu)$ are analogous.

In the offline stage, the notable computations are: (i) $2Q_a$ eigenproblems over X to form $\mathcal{B}$; (ii) $K_{\max}$ eigenproblems over X to form $\{\alpha_a(\omega) \mid \omega \in C_{K_{\max}}\}$; (iii) $K_{\max}Q_a$ inner products over X to form $\mathcal{Y}_{\rm UB}(C_{K_{\max}}) = \{y^*(\omega) \mid \omega \in C_{K_{\max}}\}$; and (iv) $JK_{\max}$ lower bound LP's of "size" $2Q_a + L$ to compute $\alpha_a^{\rm LB}(\mu)$. The total computation cost thus depends on $\mathcal{N}$. However, in practice, we can choose both J and $\mathcal{N}$ very large. (We refer the reader to Refs. 21, 46 for more details.)

5. Sampling Procedure

To achieve optimal efficiency both in the offline and online stage, we must construct a good (rapidly convergent) reduced basis space and associated basis functions; an "optimal" sampling process allows us to greatly reduce the requisite number of "truth" finite element calculations in the offline stage, and to achieve optimal efficiency in the online stage. We thus employ a greedy sampling procedure, first introduced in Ref. 53 in the context of RB methods, and placed on a more solid theoretical basis in Refs. 4 and 7.

To begin, we specify a very large, exhaustive, "training" sample of n_{train} points in $\mathcal{D}$, Ξ_{train}, and an initial (say, randomly chosen) sample $S_1 = \{\mu^1\}$. The RB greedy algorithm is then given in Algorithm 2. In actual practice, we typically terminate the greedy procedure at $N = N_{\max} \leq N^*_{\max}$ for which a prescribed error tolerance is satisfied: to wit, we define the maximum relative error

$$\delta_N^\epsilon(\mu) = \frac{\Delta_N^\epsilon(\mu)}{\left(\|u_N^\epsilon(\mu)\|_X^2 + \epsilon \frac{\alpha_c^{\text{UB}}(\mu)}{\alpha_a^{\text{UB}}(\mu)} \|p_N^\epsilon(\mu)\|_Y^2 \right)^{\frac{1}{2}}} \tag{5.1}$$

where the denominator is a (online) rapidly computable surrogate for $\|(u^\epsilon(\mu), p^\epsilon(\mu))\|_{Z,\mu,\epsilon}$. We then terminate when

$$\max_{\mu \in \Xi_{\text{train}}} \delta_N^\epsilon(\mu) \leq \delta_{\text{tol}}^{\text{RB}}.$$

6. Numerical Results

6.1. *Truth Approximation*

In this section we present numerical results for the model problem described in Section 2. For our truth discretization we take a classical P_2-P_1 (quadratic/linear) Taylor-Hood discretization (see, e.g., Ref. 51) with a total of $\mathcal{N} = 72{,}265$ velocity and pressure degrees of freedom. We present in Fig. 2 the velocity streamlines for three different parameter values $\mu = (0.1, 0.1)$, $(0.3, 0.3)$, and $(0.5, 0.5)$ computed using the standard ($\epsilon = 0$) Stokes equations. We observe that the μ-dependence over our parameter domain is clearly non-trivial.

Figure 3 plots the velocity streamlines computed using the penalty approximation with $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$, and 10^{-5}. The velocity streamlines for $\epsilon = 0$ are also plotted for comparison. This illustrates that our "truth" penalty approximation is certainly less-than-perfect, but for most engineering applications, the approximation is sufficiently accurate even for reasonably large values of ϵ. This is further illustrated in Fig. 4 which shows the error introduced by the penalty approximation as a function of the penalty parameter, ϵ, for different values of μ. Furthermore, the results in Fig. 4 clearly manifests the $O(\epsilon)$ dependence of the error predicted by (2.34).

Algorithm 2 RB Greedy Algorithm

Choose $\mu^1 \in \mathcal{D}$, $\Xi_{\text{train}} \subseteq \mathcal{D}$, $\delta_{\text{tol}}^{\text{RB}} \in (0,1)$
Set $M \leftarrow 0$, $N \leftarrow 0$, $S_N \leftarrow \{\}$, $X_N \leftarrow \{\}$, $Y_N \leftarrow \{\}$
repeat
  $N \leftarrow N+1$
  $S_N \leftarrow S_{N-1} \cup \{\mu^N\}$
  $Y_N \leftarrow Y_{N-1} \oplus \text{span}\{\, p^\epsilon(\mu^N)\,\}$
  if *Method 1* **then**
    $M \leftarrow M + (1+Q_b)N$
    $X_N \leftarrow X_{N-1} \oplus \text{span}\{\, u^\epsilon(\mu^N),\, T^k p^\epsilon(\mu^N),\ 1 \le k \le Q_b\,\}$
  else if *Method 2* **then**
    $M \leftarrow M + 2$
    $X_N \leftarrow X_{N-1} \oplus \text{span}\{\, u^\epsilon(\mu^N),\, T_{\mu^N} p^\epsilon(\mu^N)\,\}$
  end if
  for all $\mu \in \Xi_{\text{train}}$ **do**
    Compute $\left(u_N^\epsilon(\mu), p_N^\epsilon(\mu)\right)$ (see Section 3)
    Compute $\delta_N^\epsilon(\mu)$ of (5.1)
  end for
  $\mu^{N+1} = \arg \max_{\mu \in \Xi_{\text{train}}} \delta_N^\epsilon(\mu)$
until $\delta_N^\epsilon(\mu^{N+1}) < \delta_{\text{tol}}^{\text{RB}}$
$M_{\max} \leftarrow M$, $N_{\max} \leftarrow N$

6.2. *Lower and Upper Bounds for the Coercivity Constants*

We now turn to the coercivity constants required for our reduced basis procedure. We perform the SCM procedure to construct the required lower bounds for the coercivity constants. Setting $L_\alpha = 5$, $\delta_{\text{tol}}^{\text{SCM}} = 0.5$, and $J = 2000$, we obtain $K_{\max} = 8$ for $\alpha_a(\mu)$ and $K_{\max} = 4$ for $\alpha_c(\mu)$. Fig. 5 shows the lower bound $\alpha_a^{\text{LB}}(\mu)$, and upper bound $\alpha_a^{\text{UB}}(\mu)$ as a function of (a) μ_1, holding μ_2 constant; and (b) μ_2, holding μ_1 constant. Analogous results for $\alpha_c^{\text{LB}}(\mu)$ and $\alpha_c^{\text{UB}}(\mu)$ are shown in Fig. 6.

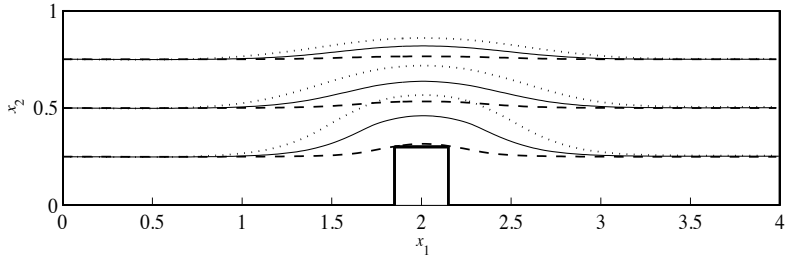

Fig. 2. Velocity streamlines for $\mu = (0.1, 0.1)$ (dashed), $(0.3, 0.3)$ (solid), and $(0.5, 0.5)$ (dotted) computed usiqng the standard Stokes equations ($\epsilon = 0$) and plotted on the reference domain Ω.

6.3. *Reduced Basis Approximation*

We now turn to the reduced basis approximation. We choose a uniformly distributed sample $\Xi_{\text{train}} \subset \mathcal{D}$, $n_{\text{train}} = 500$, and pursue the greedy sampling procedure. We then plot the maximum relative error over a test sample Ξ_{test} of 50 randomly

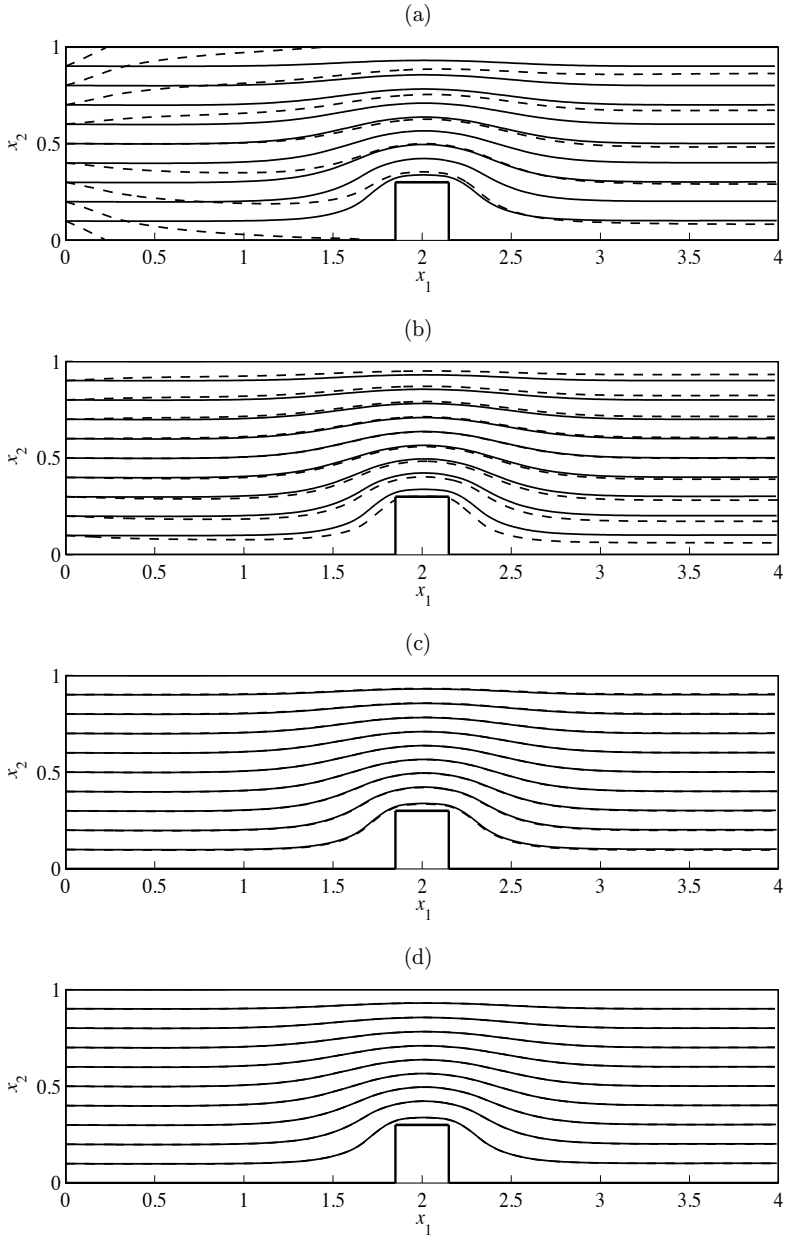

Fig. 3. Streamlines computed using (a) $\epsilon = 10^{-2}$, (b) $\epsilon = 10^{-3}$, (c) $\epsilon = 10^{-4}$, and (d) $\epsilon = 10^{-5}$. Comparison with solution for $\epsilon = 0$ (solid).

chosen parameter values. In particular, we plot in Fig. 7 the maximum relative error over all $\mu \in \Xi_{\text{test}}$ in the (a) velocity, $\|u^\epsilon(\mu) - u_N^\epsilon(\mu)\|_X / \|u^\epsilon(\mu)\|_X$, and (b) pressure, $\|p^\epsilon(\mu) - p_N^\epsilon(\mu)\|_Y / \|p^\epsilon(\mu)\|_Y$, obtained using Method 1 as a function of N for different values of ϵ. Analogous results for Method 2 are shown in Fig. 8.

The results clearly show that for both Methods 1 and 2, (i) the penalty parameter has no effect on the convergence of the reduced basis approximation; and (ii) the velocity approximation spaces are rich enough to obtain a stable approximation. Finally, we note that for this example, the error obtained using Method 1 is only slightly smaller than that using Method 2. Therefore, despite the theoretical

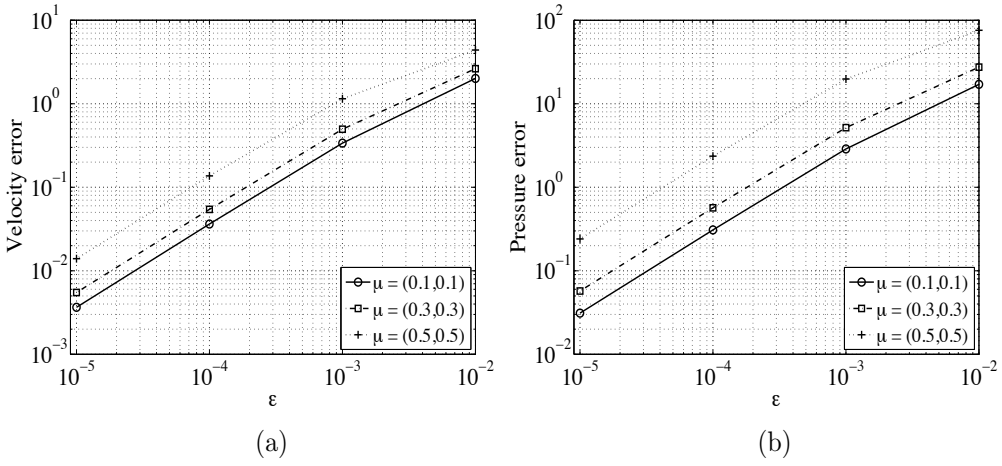

Fig. 4. Error in the (a) velocity, $\|u(\mu) - u^\epsilon(\mu)\|_X$, and (b) pressure, $\|p(\mu) - p^\epsilon(\mu)\|_Y$, introduced by the penalty approximation as a function of the penalty parameter ϵ for different values of μ.

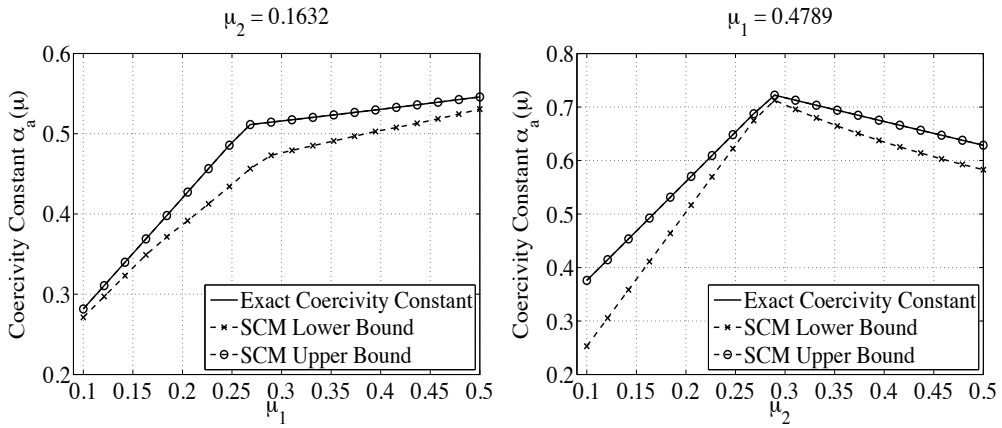

Fig. 5. SCM lower and upper bounds $\alpha_a^{\text{LB}}(\mu)$ and $\alpha_a^{\text{UB}}(\mu)$ for the coercivity constant $\alpha_a(\mu)$ with $J = 2000$, $L_\alpha = 5$, $\delta_{\text{tol}}^{\text{SCM}} = 0.5$ and $K = K_{\max} = 8$.

advantages of using Method 1, the considerably more efficient Method 2 is much better in practice.

6.4. *A Posteriori Error Bounds*

We now show in Fig. 9 the combined error $\mathrm{E}_N^\epsilon(\mu)$ and error bound $\Delta_N^\epsilon(\mu)$ normalized relative to $\|(u^\epsilon(\mu), p^\epsilon(\mu))\|_{Z,\mu,\epsilon}$. We again observe the good convergence due to

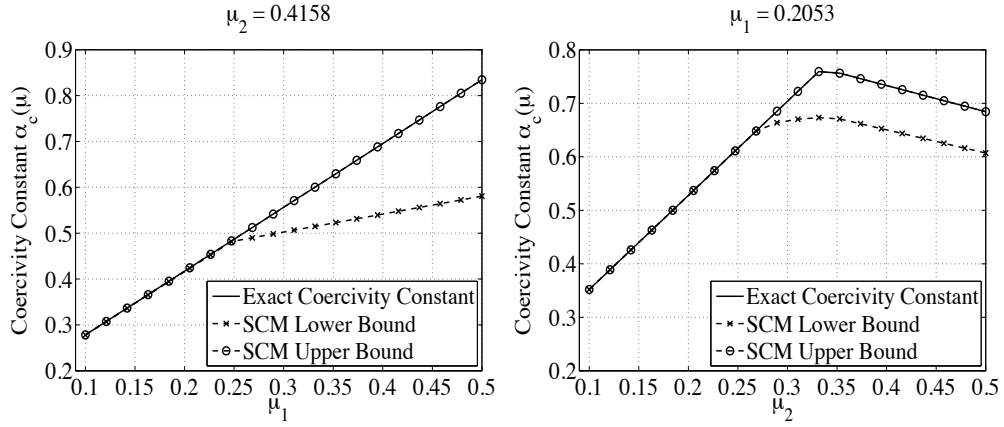

Fig. 6. SCM lower and upper bounds $\alpha_c^{\mathrm{LB}}(\mu)$ and $\alpha_c^{\mathrm{UB}}(\mu)$ for the coercivity constants $\alpha_c(\mu)$ with $J = 2000$, $L_\alpha = 5$, $\delta_{\mathrm{tol}}^{\mathrm{SCM}} = 0.5$ and $K = K_{\max} = 4$.

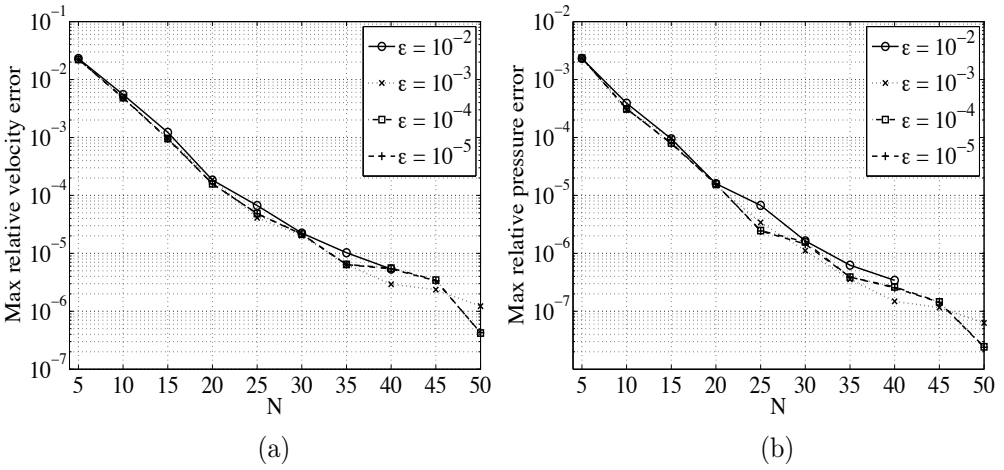

Fig. 7. Maximum relative error in the reduced basis approximation for the (a) velocity, $\|u^\epsilon(\mu) - u_N^\epsilon(\mu)\|_X / \|u^\epsilon(\mu)\|_X$, and (b) pressure, $\|p^\epsilon(\mu) - p_N^\epsilon(\mu)\|_Y / \|p^\epsilon(\mu)\|_Y$, computed using Method 1. The maximum relative error is taken over a test sample of 50 randomly chosen parameter values, and is shown as a function of N for different values of ϵ. Note that for a given N, the total dimension of the reduced basis space is given by $N + M = N + (1 + Q_b)N = 8N$.

our "optimal" reduced basis approximation spaces. We observe that for $\epsilon = 10^{-2}$, the error bounds are tight and the effectivities relatively small; however the associated truth finite element approximation may not be sufficiently accurate. As we decrease ϵ, we obtain a more accurate truth approximation, but the reduced basis error bounds become less sharp. Nevertheless, these results show that even for relatively small values of the penalty parameter, the effectivities remain reasonably small, and the dependence on ϵ is relatively weak as predicted by Corollary 4.1. To quantify this statement, we present in Tables 1(a) and 1(b) the effectivities for several values of N and ϵ for Methods 1 and 2, respectively. We note that for both Methods 1 and 2, the effectivites remain more or less constant with N. Furthermore, we show in Fig. 11 the dependence of the effectivities on the penalty parameter; we clearly observe the $\sqrt{\epsilon}$-dependence predicted by Corollary 4.1. The effects of the penalty parameter on the effectivities is thus relatively benign, and we obtain meaningful and useful bounds for reasonably small values of ϵ. In particular, the error bounds reflect the behavior of the actual error, such that the greedy algorithm is still able to choose the "right" parameter snapshots (i.e., where the error is largest).

We now discuss the Online computation times. All of the following timing tests were performed using MATLAB on a 2.66 GHz Intel Core 2 Duo processor. For comparison, we note that direct evaluation (assembly and solution) of the finite element approximation, $(u^\epsilon(\mu), p^\epsilon(\mu))$, takes 3.9 seconds.

We first consider the results for Method 1. We initially take $N = 50$, for which $M = (1 + Q_b)N = 350$; the total dimension of the reduced basis system is thus $M + N = 400$. Once the database has been loaded (this takes approximately 1.6

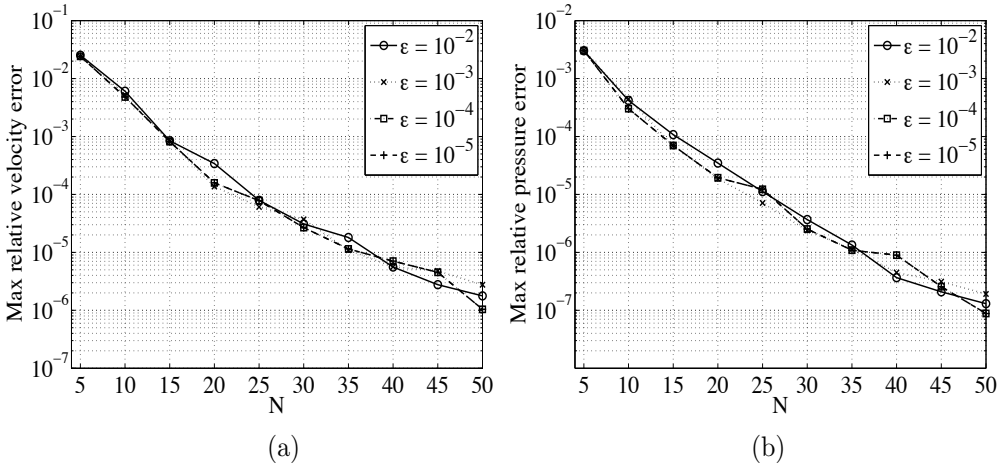

Fig. 8. Maximum relative error in the reduced basis approximation for the (a) velocity, $\|u^\epsilon(\mu) - u_N^\epsilon(\mu)\|_X / \|u^\epsilon(\mu)\|_X$, and (b) pressure, $\|p^\epsilon(\mu) - p_N^\epsilon(\mu)\|_Y / \|p^\epsilon(\mu)\|_Y$, computed using Method 2. The maximum relative error is taken over a test sample of 50 randomly chosen parameter values, and is shown as a function of N for different values of ϵ. Note that for a given N, the total dimension of the reduced basis space is given by $N + M = N + 2N = 3N$.

seconds), the Online calculation of $(u_N^\epsilon(\mu), p_N^\epsilon(\mu))$, and $\Delta_N^\epsilon(\mu)$ for any new value of μ takes 0.022 seconds and 0.23 seconds, respectively, in total roughly 15× faster than the truth approximation. Thus, even for this large value of N, we obtain significant Online savings. In practice, however, we quite often need not take such a large value of N — our rigorous error bounds allow us to choose N just large enough to obtain the desired accuracy.

We thus take $N = 25$, for which we obtain a predicted accuracy (based on our inexpensive, rigorous error bounds) of roughly 1% or better (see Fig. 9). We note that for this value of N, $M = (1 + Q_b)N = 175$ and therefore the total dimension of the reduced basis system is $M + N = 200$. Again, once the database has been loaded, the Online calculation of $(u_N^\epsilon(\mu), p_N^\epsilon(\mu))$, and $\Delta_N^\epsilon(\mu)$ for any new value of μ takes 0.006 seconds and 0.08 seconds, respectively, in total roughly 45× faster than the truth approximation.

We now turn to Method 2. We again initially take $N = 50$, for which $M = 2N = 100$, so that the total dimension of our reduced basis system is $M + N = 150$. The Online calculation of $(u_N^\epsilon(\mu), p_N^\epsilon(\mu))$ and $\Delta_N^\epsilon(\mu)$ takes 0.004 and 0.05 seconds, respectively, and is therefore roughly 70× faster than direct evaluation of the truth finite element approximation. For $N = 25$ ($M = 50$, $M + N = 75$), the Online calculation of $(u_N^\epsilon(\mu), p_N^\epsilon(\mu))$ and $\Delta_N^\epsilon(\mu)$ takes 0.001 and 0.04 seconds, respectively, and is therefore roughly 100× faster than direct evaluation of the truth approximation. Method 2 is clearly much more efficient, and the numerical results indicate that the approximation is stable despite the lack of a rigorous theoretical stability result as in Method 1.

7. Summary

We present in this paper a method based on the penalty approach for obtaining reduced basis *a posteriori* error bounds for flow problems in parametrized domains. In the proposed method, the penalty approach is used to circumvent *not* the inf-sup condition itself, but the computation of a lower bound to the global inf-sup constant previously required for rigorous error bounds. While the computation of global inf-sup lower bounds may be manageable for the Stokes problem, it becomes computationally intensive as we consider the (nonlinear) Navier-Stokes equations. At the expense of a slightly less accurate "truth" finite element approximation, the proposed approach allows us to efficiently compute reduced basis approximations and rigorous *a posteriori* error bounds for both the velocity *and* the pressure, and thus allows us to consider flow problems in parametrized domains with relative ease. The theoretical and numerical results show that the proposed error bounds are sharp, and that the effects of the penalty parameter on the effectivity of the error bounds are relative benign. We thus obtain meaningful and useful error bounds even for relatively small values of the penalty parameter.

Appendix A. Affine Decomposition

A.1. *Geometry*

The partial differential equations (2.7)-(2.9) for our model problem are defined over a parametrized domain $\tilde{\Omega}(\mu)$ illustrated in Fig. 12(a). The reduced basis method, however, requires that Ω be a *parameter-independent* domain: the basis functions or "snapshots" must be defined relative to a common spatial configuration (and finite element mesh). We must therefore transform or map our "actual" parameter-dependent domain $\tilde{\Omega}(\mu)$ to a reference domain Ω.

We thus choose a value $\mu^{\text{ref}} \in \mathcal{D}$ and define our reference domain as $\Omega = \tilde{\Omega}(\mu^{\text{ref}})$. We then introduce, for all $\mu \in \mathcal{D}$, a "reduced basis triangulation" — a domain

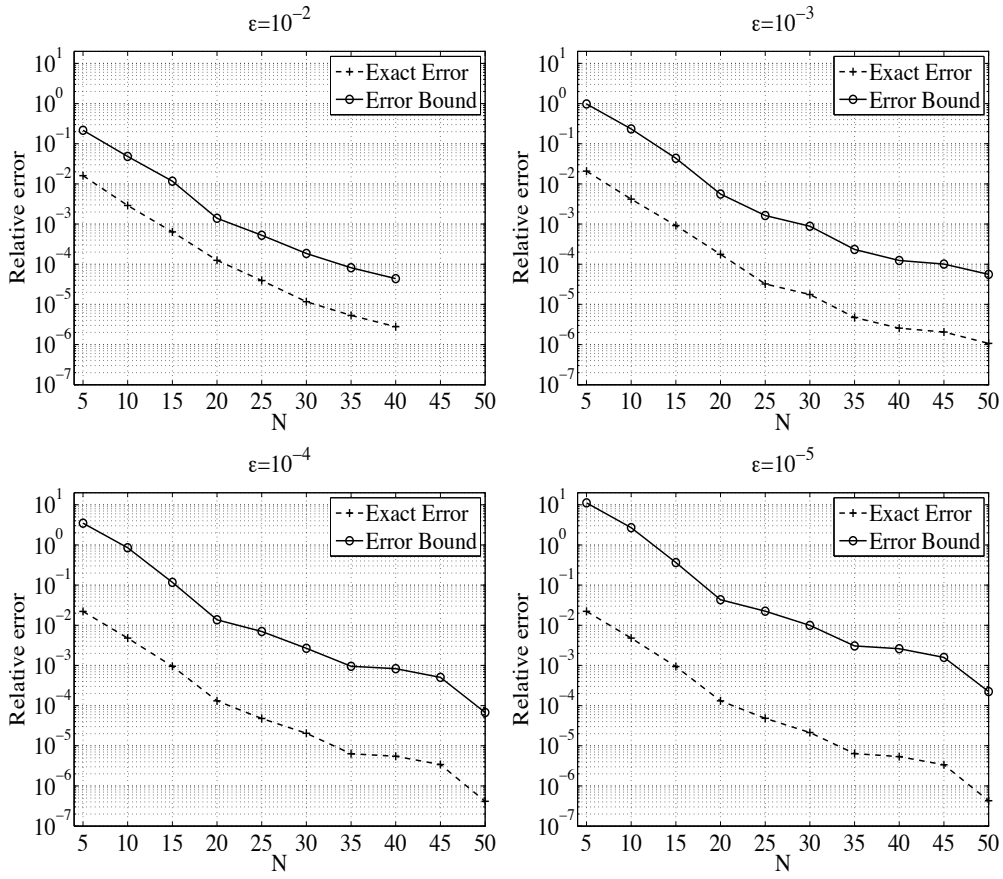

Fig. 9. Maximum relative error $\mathrm{E}_N^\epsilon(\mu)/\|(u^\epsilon(\mu), p^\epsilon(\mu))\|_{Z,\mu,\epsilon}$ and relative error bound $\Delta_N(\mu)/\|(u^\epsilon(\mu), p^\epsilon(\mu))\|_{Z,\mu,\epsilon}$ as a function of N for different values of ϵ evaluated using Method 1 (here, $M = (1 + Q_b)N = 7N$). The maximum is taken over a test sample of 50 randomly chosen parameter values.

decomposition of $\tilde{\Omega}(\mu)$,

$$\overline{\tilde{\Omega}}(\mu) = \bigcup_{k=1}^{P} \overline{\tilde{\Omega}}^k(\mu), \qquad \text{s.t. } \tilde{\Omega}^k(\mu) \cap \tilde{\Omega}^l(\mu) = \emptyset, \quad 1 \le k < l \le P, \tag{A.1}$$

from which it immediately follows from (A.1) that

$$\overline{\Omega} = \bigcup_{k=1}^{P} \overline{\Omega}^k, \qquad \text{s.t. } \Omega^k \cap \Omega^l = \emptyset, \quad 1 \le k < l \le P, \tag{A.2}$$

where $\Omega^k = \tilde{\Omega}^k(\mu^{\text{ref}})$. We further assume that

$$\overline{\tilde{\Omega}}^k(\mu) = (\mathcal{G}^k)^{-1}(\overline{\Omega}^k(\mu); \mu), \qquad 1 \le k \le P, \tag{A.3}$$

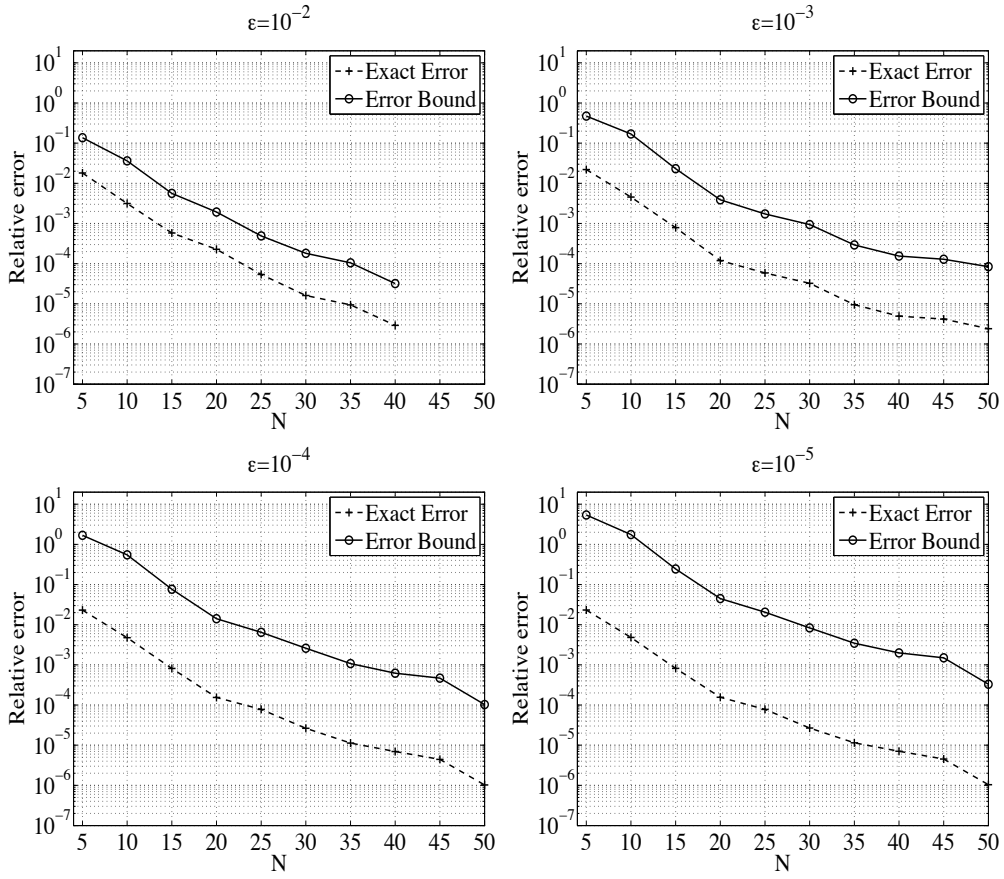

Fig. 10. Maximum relative error $\mathrm{E}_N^\epsilon(\mu)/\|(u^\epsilon(\mu), p^\epsilon(\mu))\|_{Z,\mu,\epsilon}$ and relative error bound $\Delta_N(\mu)/\|(u^\epsilon(\mu), p^\epsilon(\mu))\|_{Z,\mu,\epsilon}$ as a function of N for different values of ϵ evaluated using Method 2 (here, $M = 2N$). The maximum is taken over a test sample of 50 randomly chosen parameter values.

(a) Method 1

N	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$
5	2.530×10^1	8.074×10^1	2.554×10^2	8.077×10^2
10	2.479×10^1	7.762×10^1	2.487×10^2	7.868×10^2
15	2.677×10^1	8.887×10^1	2.336×10^2	7.440×10^2
20	2.471×10^1	7.832×10^1	2.403×10^2	7.629×10^2
25	2.297×10^1	7.049×10^1	2.605×10^2	8.258×10^2
30	2.372×10^1	8.732×10^1	2.664×10^2	8.343×10^2
35	2.210×10^1	7.962×10^1	2.482×10^2	7.890×10^2
40	2.339×10^1	7.279×10^1	2.485×10^2	7.172×10^2
45	–	7.566×10^1	2.180×10^2	6.661×10^2
50	–	7.355×10^1	2.212×10^2	7.015×10^2

(b) Method 2

N	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$
5	1.664×10^1	5.087×10^1	1.611×10^2	5.095×10^2
10	1.744×10^1	5.079×10^1	1.537×10^2	4.858×10^2
15	1.714×10^1	5.484×10^1	1.736×10^2	5.491×10^2
20	1.654×10^1	5.786×10^1	1.809×10^2	5.719×10^2
25	1.749×10^1	5.219×10^1	1.596×10^2	5.045×10^2
30	1.865×10^1	5.495×10^1	1.709×10^2	5.399×10^2
35	1.852×10^1	4.428×10^1	1.376×10^2	4.345×10^2
40	1.526×10^1	5.003×10^1	1.419×10^2	4.483×10^2
45	1.458×10^1	4.834×10^1	1.595×10^2	5.049×10^2
50	1.760×10^1	4.961×10^1	1.520×10^2	4.797×10^2

Table 1. Maximum effectivities $\eta_N^\epsilon(\mu)$ for several values of ϵ and N evaluated using (a) Method 1 and (b) Method 2. The maximum is taken over a test sample of 50 randomly chosen parameter values.

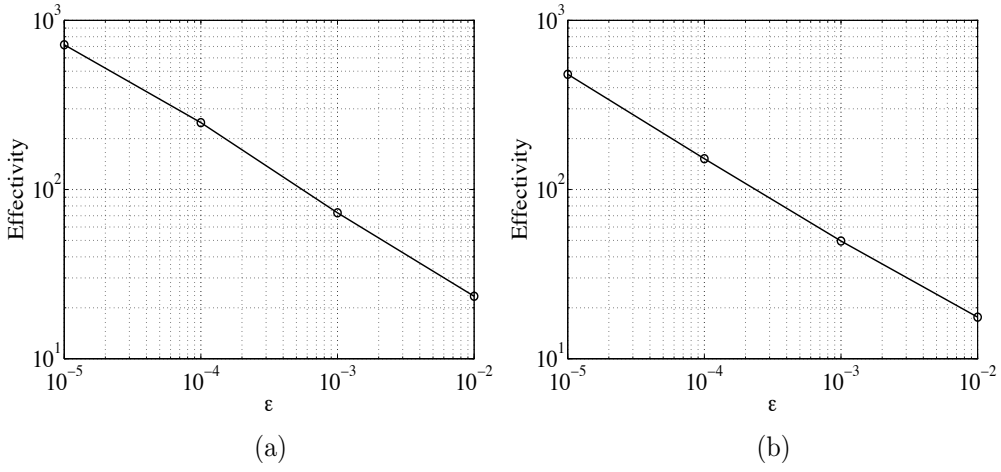

Fig. 11. Maximum effectivities $\eta_N^\epsilon(\mu)$ as a function of ϵ evaluated using (a) Method 1 ($N = 40$) and (b) Method 2 ($N = 50$) over a test sample of 50 randomly chosen parameter values.

where the $\mathcal{G}^k(\cdot;\mu) : \tilde{\Omega}^k(\mu) \to \Omega^k$ are continuous and bijective affine mappings: $(\mathcal{G}^k)^{-1}(\cdot;\mu) : \Omega^k \to \tilde{\Omega}^k(\mu)$ exists, $\mathcal{G}^k(\tilde{x};\mu) = \mathcal{G}^l(\tilde{x};\mu)$, $\forall\, \tilde{x} \in \overline{\tilde{\Omega}}^k(\mu) \cap \overline{\tilde{\Omega}}^l(\mu)$, $1 \le k < l \le P$, and

$$\mathcal{G}_i^k(\tilde{x};\mu) = \sum_{j=1}^{d} G_{ij}^k(\mu)\, \tilde{x}_j + g_i^k(\mu), \qquad 1 \le k \le P. \tag{A.4}$$

We plot in Fig. 12(b) the reference domain Ω and the corresponding reduced basis triangulation. A more detailed discussion of the affine geometry precondition can be found in Ref. 46.

We may now derive the affine representation of the required bilinear forms.

A.2. *Bilinear and Linear Forms*

We now apply standard techniques to transform the problem statement (2.7)-(2.9) to the equivalent problem statement over the reference domain (2.13)-(2.14).

We begin by defining $\tilde{X}_\mathrm{e}(\mu) = \{\tilde{v} \in (H^1(\tilde{\Omega}(\mu))^2 \,\big|\, \tilde{v}|_{\tilde{\Gamma}_0(\mu)\cup\tilde{\Gamma}_\mathrm{in}} = 0\}$ and $\tilde{Y}_\mathrm{e}(\mu) = L^2(\tilde{\Omega}(\mu))$. The weak form of the problem (2.7)-(2.9) is then: Find $\tilde{u}_\mathrm{e}^\epsilon(\mu) \in \tilde{X}_\mathrm{e}(\mu)$ and $\tilde{p}_\mathrm{e}^\epsilon(\mu) \in \tilde{Y}_\mathrm{e}(\mu)$ such that

$$\tilde{a}(\tilde{u}_\mathrm{e}^\epsilon(\mu), \tilde{v}; \mu) + \tilde{b}(\tilde{v}, \tilde{p}_\mathrm{e}^\epsilon(\mu); \mu) = \tilde{f}(\tilde{v};\mu), \quad \forall\, \tilde{v} \in \tilde{X}_\mathrm{e}(\mu), \tag{A.5}$$

$$\tilde{b}(\tilde{u}_\mathrm{e}^\epsilon(\mu), \tilde{q}; \mu) - \epsilon\, \tilde{c}(\tilde{p}_\mathrm{e}^\epsilon(\mu), \tilde{q}; \mu) = \tilde{g}(\tilde{q};\mu), \quad \forall\, \tilde{q} \in \tilde{Y}_\mathrm{e}(\mu). \tag{A.6}$$

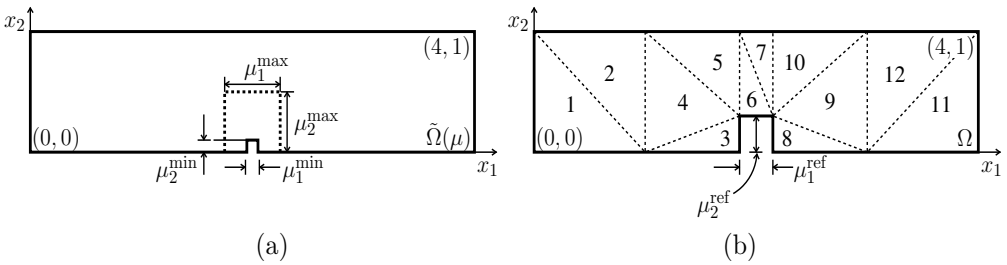

Fig. 12. (a) Parametrized domain $\tilde{\Omega}(\mu)$ for $\mu = \mu^{\max} = (0.5, 0.5)$ and $\mu = \mu^{\min} = (0.1, 0.1)$; and (b) Reference domain Ω with $\mu^{\mathrm{ref}} = (0.3, 0.3)$, and corresponding reduced basis triangulation $\overline{\Omega} = \cup_{i=1}^{P} \overline{\Omega}^k$, $P = 12$.

Here, the bilinear and linear forms are given by

$$\tilde{a}(\tilde{w},\tilde{v};\mu) = \int_{\tilde{\Omega}(\mu)} \frac{\partial \tilde{w}_i}{\partial \tilde{x}_j}\frac{\partial \tilde{v}_i}{\partial \tilde{x}_j}\, d\tilde{\Omega}(\mu), \tag{A.7}$$

$$\tilde{b}(\tilde{v},\tilde{q};\mu) = -\int_{\tilde{\Omega}(\mu)} \tilde{q}\,\frac{\partial \tilde{v}_i}{\partial \tilde{x}_i}\, d\tilde{\Omega}(\mu), \tag{A.8}$$

$$\tilde{c}(\tilde{q},\tilde{r};\mu) = \int_{\tilde{\Omega}(\mu)} \tilde{q}\,\tilde{r}\, d\tilde{\Omega}(\mu), \tag{A.9}$$

$$\tilde{f}(\tilde{v};\mu) = -\int_{\tilde{\Omega}(\mu)} \frac{\partial \tilde{u}_{\mathrm{L}\,i}}{\partial \tilde{x}_j}\frac{\partial \tilde{v}_i}{\partial \tilde{x}_j}\, d\tilde{\Omega}(\mu), \tag{A.10}$$

$$\tilde{g}(\tilde{q};\mu) = \int_{\tilde{\Omega}(\mu)} \tilde{q}\,\frac{\partial \tilde{u}_{\mathrm{L}\,i}}{\partial \tilde{x}_i}\, d\tilde{\Omega}(\mu), \tag{A.11}$$

for all $\tilde{w},\tilde{v} \in \tilde{X}_{\mathrm{e}}(\mu)$ and $\tilde{q},\tilde{r} \in \tilde{Y}_{\mathrm{e}}(\mu)$.

Applying our affine decomposition and associated mappings, we note that for $\tilde{x} = (\mathcal{G}^k)^{-1}(x;\mu) \in \tilde{\Omega}^k(\mu)$, we have $\partial/\partial\tilde{x}_i = G^k_{ji}\partial/\partial x_j$, and $d\tilde{\Omega}(\mu) = \left|\det\left(G^k(\mu)\right)^{-1}\right| d\Omega$. For functions $\tilde{v} \in \tilde{X}_{\mathrm{e}}(\mu)$, $\tilde{q} \in \tilde{Y}_{\mathrm{e}}(\mu)$ we consider the functions $v \in X_{\mathrm{e}}$, $q \in Y_{\mathrm{e}}$ defined by $v(x) = \tilde{v}(\tilde{x})$ and $q(x) = \tilde{q}(\tilde{x})$. The above model problem is then equivalent to: Find $u^\epsilon_{\mathrm{e}}(\mu) \in X_{\mathrm{e}}$ and $p^\epsilon_{\mathrm{e}}(\mu) \in Y_{\mathrm{e}}$ such that

$$a(u^\epsilon_{\mathrm{e}}(\mu),v;\mu) + b(v,p^\epsilon_{\mathrm{e}}(\mu);\mu) = f(v;\mu), \quad \forall\, v \in X_{\mathrm{e}}, \tag{A.12}$$

$$b(u^\epsilon_{\mathrm{e}}(\mu),q;\mu) - \epsilon\, c(p^\epsilon_{\mathrm{e}}(\mu),q;\mu) = g(q;\mu), \quad \forall\, q \in Y_{\mathrm{e}}. \tag{A.13}$$

Here, the bilinear and linear forms are given by

$$a(w,v;\mu) = \sum_{s=1}^{P} G^s_{kj}(\mu)G^s_{lj}(\mu)J^s(\mu)\int_{\Omega^s} \frac{\partial w_i}{\partial x_k}\frac{\partial v_i}{\partial x_l}\, d\Omega, \tag{A.14}$$

$$b(v,q;\mu) = -\sum_{s=1}^{P} G^s_{ji}(\mu)J^s(\mu)\int_{\Omega^s} q\,\frac{\partial v_i}{\partial x_j}\, d\Omega, \tag{A.15}$$

$$c(q,r;\mu) = \sum_{s=1}^{P} J^s(\mu)\int_{\Omega^s} q\,r\, d\Omega, \tag{A.16}$$

$$f(v;\mu) = -\sum_{s=1}^{P} G^s_{kj}(\mu)G^s_{lj}(\mu)J^s(\mu)\int_{\Omega^s} \frac{\partial u_{\mathrm{L}\,i}}{\partial x_k}\frac{\partial v_i}{\partial x_l}\, d\Omega, \tag{A.17}$$

$$g(q;\mu) = \sum_{s=1}^{P} G^s_{ji}(\mu)J^s(\mu)\int_{\Omega^s} q\,\frac{\partial u_{\mathrm{L}\,i}}{\partial x_j}\, d\Omega, \tag{A.18}$$

for all $w,v \in X_{\mathrm{e}}$ and $q,r \in Y_{\mathrm{e}}$, where $J^s(\mu) = \left|\det\left(G^s(\mu)\right)^{-1}\right|$. Equations (A.14)-

(A.18) may then be written succinctly as

$$a(w, v; \mu) = \sum_{k=1}^{Q_a} \Theta_a^k(\mu)\, a^k(w, v), \tag{A.19}$$

$$b(v, q; \mu) = \sum_{k=1}^{Q_b} \Theta_b^k(\mu)\, b^k(v, q), \tag{A.20}$$

$$c(q, r; \mu) = \sum_{k=1}^{Q_c} \Theta_c^k(\mu)\, c^k(q, r), \tag{A.21}$$

$$f(v; \mu) = \sum_{k=1}^{Q_f} \Theta_f^k(\mu)\, f^k(v), \tag{A.22}$$

$$g(q; \mu) = \sum_{k=1}^{Q_g} \Theta_g^k(\mu)\, g^k(q), \tag{A.23}$$

for all $w, v \in X_\mathrm{e}$ and $q, r \in Y_\mathrm{e}$; we thus satisfy our assumption of affine parameter dependence. For our model problem, the affine decomposition is obtained with $Q_a = 10$, $Q_b = 6$, $Q_c = 5$, $Q_f = 1$, and $Q_g = 1$.

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