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# Reduced Basis *A Posteriori* Error Bounds for the Instationary Stokes Equations: A Penalty Approach<sup>\*</sup>

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**Abstract:** We present reduced basis approximations and associated rigorous *a posteriori* error bounds for the instationary Stokes equations. The proposed method is an extension of the penalty approach introduced in Gerner and Veroy (2011b) to the time-dependent setting: The introduction of a penalty term enables us to develop *a posteriori* error bounds that do not rely on the expensive calculation of inf-sup stability constants and that are thus computationally very efficient. Numerical results show the rapid convergence of reduced basis approximations and demonstrate the effects of the penalty parameter.

*Keywords:* Saddle point problem; Stokes equations; incompressible fluid flow; model order reduction; reduced basis method; penalty; *a posteriori* error bounds

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## 1. INTRODUCTION

In many engineering applications, it is crucial to understand the effects of geometry variations on a flow system. When the objective is to optimize, control, or characterize the system, classical discretization techniques such as finite element, finite volume, or finite difference methods are generally too expensive. As it can often capture the system behavior at significantly less cost, reduced order modelling for fluid flow has received considerable attention. Our particular approach is the certified reduced basis method, in which the emphasis is on rigorous *a posteriori* error bounds and adaptive sampling procedures to provide rapidly convergent and computationally efficient approximations.

The RB method exploits the parametric structure of the problem to construct rapidly convergent and computationally efficient approximations equipped with rigorous error bounds. Built upon a high-fidelity “truth” discretization, the RB approximation is defined as the Galerkin projection of the “truth” model problem onto a low-dimensional approximation space that focuses on the solution manifold induced by the parametrized partial differential equation.

We here investigate low-Reynolds number fluid flow in parametrized domains. Although reduced basis (RB) methods are well-developed for several classes of partial differential equations [Grepl and Patera (2005); Prud’homme et al. (2002); Rozza et al. (2008)], incompressible fluid flow problems involving parametrized domains pose additional difficulties that have not been fully addressed: Parameter-dependent constraints in the Stokes and Navier–Stokes equations cause complications not only

in the choice of stable RB approximation spaces [Gerner and Veroy (2011a); Rozza and Veroy (2007)] but also in the construction of rigorous and computationally efficient *a posteriori* error bounds [Debaris and Rozza (2009); Gerner and Veroy (2011a); Knezevic et al. (2011)].

The analysis of the Stokes equations is often performed as a stepping stone for the more general Navier–Stokes equations but nevertheless remains relevant in many engineering applications [see, e.g., Stone et al. (2004)]. In Gerner and Veroy (2011b), we introduced a penalty approach that circumvents the expensive computation of stability inf-sup constants. Geometry variations are admitted with relative ease but at the expense of an additional error in the “truth” approximation upon which the RB approximation is built.

In this paper, we extend the approach presented in Gerner and Veroy (2011b) to time-dependent problems. We develop rigorous upper bounds for the errors in the RB approximations, which shall then be analyzed with respect to sharpness. As in the stationary case in Gerner and Veroy (2011b), the introduction of the penalty term enables us to derive *a posteriori* RB error bounds that do not require the very expensive calculation of lower bounds to inf-sup stability constants. They only depend on comparably inexpensive lower bounds to coercivity constants associated with the diffusion and penalty terms and are thus computationally very efficient. However, the error bounds also depend on the penalty parameter  $\varepsilon$ : Effectivities increase as we approach the non-penalized problem ( $\varepsilon = 0$ ). Finally, RB approximation and error bounds are intimately linked through a POD greedy approach, in which the error bounds are used to construct the subsequent approximation spaces more optimally.

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The method is applied to a Stokes flow in a two-dimensional microchannel with a parametrized rectangular obstacle. Numerical results demonstrate that the RB approximations converge rapidly, the developed rigorous *a posteriori* RB error bounds are sharp, and that the effects of the penalty parameter on the effectivity of the error bounds are relatively benign.

## 2. GENERAL PROBLEM STATEMENT

### 2.1 Strong Formulation on the Parametrized Domain

We consider a Stokes flow in a two-dimensional channel with an obstacle as illustrated as in Gerner and Veroy (2011b). The problem depends on two geometric parameters  $\mu \equiv (\mu^1, \mu^2)$  in  $\mathcal{D} \equiv [0.1, 0.5]^2$ , representing width  $\mu^1$  and height  $\mu^2$  of the obstacle  $\mathcal{O}(\mu) \equiv ([\frac{2-\mu^1}{2}, \frac{2+\mu^1}{2}] \times [0, \mu^2])$ . We denote the physical domain by  $\bar{\Omega} \equiv \Omega(\mu) \equiv ([0, 4] \times [0, 1]) \setminus \mathcal{O}(\mu)$  with its boundary  $\bar{\Gamma} \equiv \Gamma(\mu)$ . We assume fully-developed flow conditions with a parabolic velocity profile on the inflow boundary  $\Gamma_{\text{in}} \equiv \{0\} \times [0, 1]$ , natural outflow conditions on  $\Gamma_{\text{out}} \equiv \{4\} \times [0, 1]$ , and no-slip velocity conditions on the channel walls and obstacle boundary  $\bar{\Gamma}_0 \equiv \Gamma_0(\mu) \equiv \Gamma(\mu) \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}})$ .

For the physical domain  $\bar{\Omega}$  and a given time-interval  $[0, T]$ ,  $T > 0$ , we now seek to find the (inhomogeneous) velocity  $\bar{u}_{\text{inh}} : \bar{\Omega} \times (0, T) \rightarrow \mathbb{R}^2$  and the pressure  $\bar{p} : \bar{\Omega} \times (0, T) \rightarrow \mathbb{R}$  such that

$$\frac{\partial \bar{u}_{\text{inh}}}{\partial t} - \bar{\Delta} \bar{u}_{\text{inh}} + \bar{\nabla} \bar{p} = \bar{0} \text{ in } \bar{\Omega} \times (0, T), \quad (1)$$

$$-\bar{\nabla} \cdot \bar{u}_{\text{inh}} = 0 \text{ in } \bar{\Omega} \times (0, T), \quad (2)$$

with boundary conditions

$$\bar{u}_{\text{inh}} = \bar{u}_L \text{ on } (\Gamma_{\text{in}} \cup \bar{\Gamma}_0) \times (0, T), \quad (3)$$

$$\frac{\partial \bar{u}_{\text{inh}}}{\partial \bar{n}} = \bar{p} \bar{n} \text{ on } \Gamma_{\text{out}} \times (0, T), \quad (4)$$

and subject to initial conditions  $\bar{u}_{\text{inh}}(\cdot, 0) = \bar{0}$ ; here,  $\bar{n}$  denotes the unit outward normal and we choose the lifting function  $\bar{u}_L : \bar{\Omega} \times (0, T) \rightarrow \mathbb{R}^2$ , with  $\bar{u}_L(\bar{x}, t) \equiv (16t(1-t)\bar{x}_2(1-\bar{x}_2)(1-\bar{x}_1)^2, 0)$  for  $\bar{x} \in \Omega_L \equiv (0, 1)^2$ ,  $\bar{u}_L \equiv \bar{0}$  elsewhere. The velocity may then be decomposed into  $\bar{u}_{\text{inh}} = \bar{u}_L + \bar{u}$ , where  $\bar{u} : \bar{\Omega} \times (0, T) \rightarrow \mathbb{R}^2$  satisfies homogeneous Dirichlet boundary conditions on  $\Gamma_{\text{in}} \cup \bar{\Gamma}_0$ .

We now introduce a penalty term to the continuity equation (2),

$$\bar{\nabla} \cdot \bar{u}_{\text{inh}}^\varepsilon = -\varepsilon \bar{p}^\varepsilon \text{ in } \bar{\Omega} \times (0, T), \quad (5)$$

for some small penalty parameter  $\varepsilon > 0$ , essentially approximating the fluid as *almost* incompressible. The governing equations for the (homogeneous) velocity  $\bar{u}^\varepsilon$  and the pressure  $\bar{p}^\varepsilon$  are then

$$\frac{\partial \bar{u}^\varepsilon}{\partial t} - \bar{\Delta} \bar{u}^\varepsilon + \bar{\nabla} \bar{p}^\varepsilon = \bar{f} \text{ in } \bar{\Omega} \times (0, T), \quad (6)$$

$$-\bar{\nabla} \cdot \bar{u}^\varepsilon - \varepsilon \bar{p}^\varepsilon = \bar{g} \text{ in } \bar{\Omega} \times (0, T), \quad (7)$$

where  $\bar{f} \equiv \bar{\Delta} \bar{u}_L - \frac{\partial \bar{u}_L}{\partial t}$  and  $\bar{g} \equiv \bar{\nabla} \cdot \bar{u}_L$ , with boundary and initial conditions

$$\bar{u}^\varepsilon|_{(\Gamma_{\text{in}} \cup \bar{\Gamma}_0) \times (0, T)} = \bar{0}, \quad \left( \frac{\partial \bar{u}^\varepsilon}{\partial \bar{n}} - \bar{p}^\varepsilon \bar{n} \right) \Big|_{\Gamma_{\text{out}} \times (0, T)} = \bar{0}, \quad (8)$$

$$\bar{u}^\varepsilon(\cdot, 0) = \bar{0}. \quad (9)$$

### 2.2 Variational Formulation on a Reference Domain

We define the function spaces  $\bar{V} \equiv \{ \bar{v} \in (H^1(\bar{\Omega}))^2 \mid \bar{v}|_{\bar{\Gamma}_0 \cup \Gamma_{\text{in}}} = \bar{0} \}$ ,  $\bar{Q} \equiv L^2(\bar{\Omega})$ , and  $\bar{L} \equiv (L^2(\bar{\Omega}))^2 = \bar{Q} \times \bar{Q}$ ; for any details on Sobolev and associated Bochner spaces, we refer to, e.g., Adams and Fournier (2003). The  $L^2$ -inner product is denoted by  $(\cdot, \cdot)$ . We then consider the following variational formulation of (6)–(9): For any given  $\mu \in \mathcal{D}$  and  $\varepsilon > 0$ , we find  $\bar{u}^\varepsilon \in L^2(0, T; \bar{V})$ ,  $\frac{d}{dt} \bar{u}^\varepsilon \in L^2(0, T; \bar{L})$ , and  $\bar{p}^\varepsilon \in L^2(0, T; \bar{Q})$  such that a.e. in  $t \in (0, T)$ ,

$$\left( \frac{d}{dt} \bar{u}^\varepsilon, \bar{v} \right) + \bar{a}(\bar{u}^\varepsilon, \bar{v}) + \bar{b}(\bar{v}, \bar{p}^\varepsilon) = (\bar{f}, \bar{v}), \quad \forall \bar{v} \in \bar{V}, \quad (10)$$

$$\bar{b}(\bar{u}^\varepsilon, \bar{q}) - \varepsilon (\bar{p}^\varepsilon, \bar{q}) = (\bar{g}, \bar{q}), \quad \forall \bar{q} \in \bar{Q}, \quad (11)$$

$$\bar{u}^\varepsilon(0) = \bar{0}. \quad (12)$$

By our choice of  $\bar{u}_L \in C^\infty(0, T; (H^2(\bar{\Omega}))^2)$ , it is in particular  $\bar{f} \in C^\infty(0, T; \bar{L})$  and  $\bar{g} \in C^\infty(0, T; H^1(\bar{\Omega}))$  for all  $\mu \in \mathcal{D}$ . For any parameter  $\mu \in \mathcal{D}$ , the above problem (10)–(12) is well-posed and has a unique solution [see, e.g., Hebeker (1982); Temam (1979)].

The problem statement over the physical domain  $\bar{\Omega}$  is now transformed to an equivalent problem posed over a *parameter-independent* reference domain  $\Omega$ . To this end,  $\bar{\Omega} = \Omega(\mu)$  is traced back to  $\Omega \equiv \Omega(\mu_{\text{ref}})$  by a continuous, piecewise affine mapping; we denote the boundary of  $\Omega$  by  $\Gamma \equiv \Gamma(\mu_{\text{ref}})$ ,  $\Gamma_0 \equiv \Gamma_0(\mu_{\text{ref}})$ . For further details on the mapping procedure for this particular model problem, we refer to Gerner and Veroy (2011b); for more general problems, we refer to Rozza et al. (2008). The function spaces associated with  $\Omega$  are denoted by  $V$ ,  $Q$ , and  $L$ , respectively. The transformed problem of (10)–(12) then reads: For any  $\mu \in \mathcal{D}$  and  $\varepsilon > 0$ , we find  $u^\varepsilon(\mu) \in L^2(0, T; V)$ ,  $\frac{d}{dt} u^\varepsilon(\mu) \in L^2(0, T; L)$ , and  $p^\varepsilon(\mu) \in L^2(0, T; Q)$  such that a.e. in  $t \in (0, T)$ ,

$$m\left(\frac{d}{dt} u^\varepsilon(\mu), v; \mu\right) + a(u^\varepsilon(\mu), v; \mu) + b(v, p^\varepsilon(\mu); \mu) = (f, v), \quad \forall v \in V, \quad (13)$$

$$b(u^\varepsilon(\mu), q; \mu) - \varepsilon c(p^\varepsilon(\mu), q; \mu) = (g, q), \quad \forall q \in Q, \quad (14)$$

$$u^\varepsilon(0; \mu) = \bar{0}. \quad (15)$$

Here,  $m(\cdot, \cdot; \mu) : L \times L \rightarrow \mathbb{R}$ ,  $a(\cdot, \cdot; \mu) : V \times V \rightarrow \mathbb{R}$ ,  $b(\cdot, \cdot; \mu) : V \times Q \rightarrow \mathbb{R}$ , and  $c(\cdot, \cdot; \mu) : Q \times Q \rightarrow \mathbb{R}$  are the continuous bilinear forms associated with the mass term, the diffusion term, the incompressibility constraint, and the penalty term, respectively, posed over the reference domain  $\Omega$ ; they all in particular depend affinely on the parameter  $\mu$ . The bilinear forms  $m(\cdot, \cdot; \mu)$ ,  $a(\cdot, \cdot; \mu)$ , and  $c(\cdot, \cdot; \mu)$  are moreover symmetric and coercive.

### 2.3 Truth Approximation

The RB approximation is built upon a high-fidelity “truth” finite element approximation. To this end, we introduce a regular triangulation  $\mathcal{T}_\Omega$  of  $\Omega$  and denote by  $X \times Y \subset V \times Q$  the standard conforming  $\mathbb{P}_2$ – $\mathbb{P}_1$  (quadratic-linear) Taylor–Hood finite element approximation subspace over  $\mathcal{T}$ . Furthermore, we divide the time-interval  $[0, T]$  into  $K$  subintervals of equal length  $\Delta t \equiv T/K$ , and define  $t^k \equiv k\Delta t$ ,  $1 \leq k \leq K$ .

Using a fully implicit time-scheme, our “truth” discretization of (13)–(15) is now defined as follows: For any given

$\mu \in \mathcal{D}$ , we find  $u^{\varepsilon,k}(\mu) \in X$  and  $p^{\varepsilon,k}(\mu) \in Y$ ,  $0 \leq k \leq K$ , such that

$$\frac{1}{\Delta t} m(u^{\varepsilon,k}(\mu) - u^{\varepsilon,k-1}(\mu), v; \mu) + a(u^{\varepsilon,k}(\mu), v; \mu) + b(v, p^{\varepsilon,k}(\mu); \mu) = (f^k, v), \quad \forall v \in X, \quad (16)$$

$$b(u^{\varepsilon,k}(\mu), q; \mu) - \varepsilon c(p^{\varepsilon,k}(\mu), q; \mu) = (g^k, q), \quad \forall q \in Y, \quad (17)$$

$$u^{\varepsilon,0}(\mu) = 0, \quad (18)$$

for all  $1 \leq k \leq K$ , where  $f^k \equiv f(t^k)$ ,  $g^k \equiv g(t^k)$ ,  $1 \leq k \leq K$ . By the continuity of all involved bilinear and linear forms and the coercivity of  $m(\cdot, \cdot; \mu)$ ,  $a(\cdot, \cdot; \mu)$ , and  $c(\cdot, \cdot; \mu)$ , the discrete problem (16)–(18) is well-posed [Brezzi and Fortin (1991), p. 45, Proposition 1.4].

As we need them later to formulate *a posteriori* RB error bounds, we here introduce the coercivity constants

$$\alpha_a(\mu) \equiv \inf_{v \in X} \frac{a(v, v; \mu)}{\|v\|_X^2} > 0, \quad \forall \mu \in \mathcal{D}, \quad (19)$$

$$\alpha_c(\mu) \equiv \inf_{q \in Y} \frac{c(q, q; \mu)}{\|q\|_Y^2} > 0, \quad \forall \mu \in \mathcal{D}. \quad (20)$$

### 3. REDUCED BASIS METHOD

#### 3.1 Reduced Basis Approximation

We now turn to the RB approximation. Let us suppose for now that we are given a set of hierarchical, low-dimensional RB approximation subspaces  $X_N \subset X$  and  $Y_N \subset Y$ ,  $N \in [1, N_{\max}]$ ; let  $N$  denote the dimension of  $X_N \times Y_N$ . The RB approximation is then given as the Galerkin projection with respect to the truth problem (16)–(18) onto these low-dimensional subspaces. For  $\mu \in \mathcal{D}$ , we find  $u_N^{\varepsilon,k}(\mu) \in X_N$  and  $p_N^{\varepsilon,k}(\mu) \in Y_N$ ,  $0 \leq k \leq K$ , such that

$$\frac{1}{\Delta t} m(u_N^{\varepsilon,k}(\mu) - u_N^{\varepsilon,k-1}(\mu), v; \mu) + a(u_N^{\varepsilon,k}(\mu), v; \mu) + b(v, p_N^{\varepsilon,k}(\mu); \mu) = (f^k, v), \quad \forall v \in X_N, \quad (21)$$

$$b(u_N^{\varepsilon,k}(\mu), q; \mu) - \varepsilon c(p_N^{\varepsilon,k}(\mu), q; \mu) = (g^k, q), \quad \forall q \in Y_N, \quad (22)$$

$$u_N^{\varepsilon,0}(\mu) = 0, \quad (23)$$

for all  $1 \leq k \leq K$ . The problem is then well-posed by analogous arguments as in §2.3.

#### 3.2 A Posteriori Error Estimation

We aim to develop not only efficient reduced order approximations, but also *a posteriori* error estimates that are rigorous, sharp, and computationally inexpensive. As in the stationary case [see Gerner and Veroy (2011b)], the introduction of the penalty term (5) enables us to derive *a posteriori* error bounds that do not require the very expensive calculation of lower bounds to inf-sup stability constants. They only depend on comparably inexpensive lower bounds to the coercivity constants (19), (20) and are thus computationally very efficient.

For any  $\mu \in \mathcal{D}$ ,  $1 \leq k \leq K$ , we now consider the errors

$$\begin{aligned} e_N^{u,k}(\mu) &\equiv u^{\varepsilon,k}(\mu) - u_N^{\varepsilon,k}(\mu) \in X, \\ e_N^{p,k}(\mu) &\equiv p^{\varepsilon,k}(\mu) - p_N^{\varepsilon,k}(\mu) \in Y, \\ e_N^{\varepsilon,k}(\mu) &\equiv (e_N^{u,k}(\mu), e_N^{p,k}(\mu)) \in Z, \end{aligned} \quad (24)$$

in the RB velocity and pressure approximations, which shall be measured in the “spatio-temporal” energy norm

$$\begin{aligned} \|\| e_N^{\varepsilon,k}(\mu) \|\|_{\mu,\varepsilon}^2 &\equiv m(e_N^{u,k}(\mu), e_N^{u,k}(\mu); \mu) \\ &+ \Delta t \sum_{j=1}^k \left[ a(e_N^{u,j}(\mu), e_N^{u,j}(\mu); \mu) + \varepsilon c(e_N^{p,j}(\mu), e_N^{p,j}(\mu); \mu) \right]. \end{aligned} \quad (25)$$

To construct RB error bounds  $\Delta_N^{\varepsilon,k}(\mu)$  satisfying

$$\|\| e_N^{\varepsilon,k}(\mu) \|\|_{\mu,\varepsilon} \leq \Delta_N^{\varepsilon,k}(\mu), \quad (26)$$

for all  $\mu \in \mathcal{D}$ ,  $1 \leq k \leq K$ , we first have to introduce two sets of ingredients. The first set of ingredients consists of the above mentioned lower (and upper) bounds to the truth coercivity constants (19) and (20),

$$\begin{aligned} \alpha_a^{\text{LB}}(\mu) \leq \alpha_a(\mu) \leq \alpha_a^{\text{UB}}(\mu), \\ \alpha_c^{\text{LB}}(\mu) \leq \alpha_c(\mu) \leq \alpha_c^{\text{UB}}(\mu), \end{aligned} \quad \forall \mu \in \mathcal{D}. \quad (27)$$

The second set of ingredients consists of the dual norms of the residuals  $r_N^{1,k}(\cdot; \mu) \in X'$  and  $r_N^{2,k}(\cdot; \mu) \in Y'$  given by

$$\begin{aligned} r_N^{1,k}(v; \mu) &\equiv (f^k, v) - \frac{1}{\Delta t} m(u_N^{\varepsilon,k}(\mu) - u_N^{\varepsilon,k-1}(\mu), v; \mu) \\ &- a(u_N^{\varepsilon,k}(\mu), v; \mu) - b(v, p_N^{\varepsilon,k}(\mu); \mu), \quad \forall v \in X, \end{aligned} \quad (28)$$

$$\begin{aligned} r_N^{2,k}(q; \mu) &\equiv (g^k, q) - b(u_N^{\varepsilon,k}(\mu), q; \mu) \\ &+ \varepsilon c(p_N^{\varepsilon,k}(\mu), q; \mu), \quad \forall q \in Y. \end{aligned} \quad (29)$$

We can now state:

*Proposition 1.* For any given  $\mu \in \mathcal{D}$ ,  $\Delta_N^{\varepsilon,k}(\mu)$  defined as

$$\Delta_N^{\varepsilon,k}(\mu) \equiv \left( \Delta t \sum_{j=1}^k \frac{\|r_N^{1,k}(\cdot; \mu)\|_{X'}^2}{\alpha_a^{\text{LB}}(\mu)} + \frac{\|r_N^{2,k}(\cdot; \mu)\|_{Y'}^2}{\varepsilon \alpha_c^{\text{LB}}(\mu)} \right)^{1/2} \quad (30)$$

satisfies (26) for any  $N \in [1, N_{\max}]$ .

**Proof.** Let  $\mu$  be any parameter in  $\mathcal{D}$ ,  $k \in [1, K]$ , and  $N \in [1, N_{\max}]$ . For clarity of exposition, we suppress all  $\mu$ -dependence in what follows.

From (28), (29) and (16), (17), the errors  $e_N^{u,j}$  and  $e_N^{p,j}$ ,  $1 \leq j \leq k$ , satisfy the equations

$$\begin{aligned} \frac{1}{\Delta t} m(e_N^{u,j} - e_N^{u,j-1}, v) + a(e_N^{u,j}, v) + b(v, e_N^{p,j}) &= r_N^{1,j}(v), \\ b(e_N^{u,j}, q) - \varepsilon c(e_N^{p,j}, q) &= r_N^{2,j}(q), \end{aligned}$$

for all  $v \in X$ ,  $q \in Y$ . Setting  $v = e_N^{u,j}$ ,  $q = e_N^{p,j}$ , we obtain

$$\begin{aligned} \frac{1}{\Delta t} m(e_N^{u,j} - e_N^{u,j-1}, e_N^{u,j}) + a(e_N^{u,j}, e_N^{u,j}) + b(e_N^{u,j}, e_N^{p,j}) \\ = r_N^{1,j}(e_N^{u,j}), \end{aligned}$$

$$b(e_N^{u,j}, e_N^{p,j}) - \varepsilon c(e_N^{p,j}, e_N^{p,j}) = r_N^{2,j}(e_N^{p,j}), \quad \forall 1 \leq j \leq k;$$

subtracting the second from the first equation then yields

$$\frac{1}{\Delta t} m(e_N^{u,j}, e_N^{u,j}) + a(e_N^{u,j}, e_N^{u,j}) + \varepsilon c(e_N^{p,j}, e_N^{p,j}) \quad (31)$$

$$= \frac{1}{\Delta t} m(e_N^{u,j-1}, e_N^{u,j}) + r_N^{1,j}(e_N^{u,j}) - r_N^{2,j}(e_N^{p,j}), \quad (32)$$

for all  $1 \leq j \leq k$ .

We now consider the right-hand-side (32). From the Cauchy-Schwarz and Young's inequality, it follows that

$$\begin{aligned} m(e_N^{u,j-1}, e_N^{u,j}) &\leq \sqrt{m(e_N^{u,j-1}, e_N^{u,j-1})} \sqrt{m(e_N^{u,j}, e_N^{u,j})} \\ &\leq \frac{1}{2} \left( m(e_N^{u,j-1}, e_N^{u,j-1}) + m(e_N^{u,j}, e_N^{u,j}) \right), \end{aligned}$$

for all  $1 \leq j \leq k$ . Again using Young's inequality with  $\rho_1, \rho_2 > 0$ , it is

$$\begin{aligned} r_N^{1,j}(e_N^{u,j}) - r_N^{2,j}(e_N^{p,j}) &\leq \|r_N^{1,j}\|_{X'} \|e_N^{u,j}\|_X + \|r_N^{2,j}\|_{Y'} \|e_N^{p,j}\|_Y \\ &\leq \frac{1}{2} \left( \frac{1}{\rho_1} \|r_N^{1,j}\|_{X'}^2 + \rho_1 \|e_N^{u,j}\|_X^2 + \frac{1}{\rho_2} \|r_N^{2,j}\|_{Y'}^2 + \rho_2 \|e_N^{p,j}\|_Y^2 \right), \end{aligned}$$

for all  $1 \leq j \leq k$ .

Inserting both estimates in (31), (32), we have

$$\begin{aligned} &\frac{1}{\Delta t} \left( m(e_N^{u,j}, e_N^{u,j}) - m(e_N^{u,j-1}, e_N^{u,j-1}) \right) + 2a(e_N^{u,j}, e_N^{u,j}) \\ &\quad - \rho_1 \|e_N^{u,j}\|_X^2 + 2\varepsilon c(e_N^{p,j}, e_N^{p,j}) - \rho_2 \|e_N^{p,j}\|_Y^2 \\ &\leq \frac{1}{\rho_1} \|r_N^{1,j}\|_{X'}^2 + \frac{1}{\rho_2} \|r_N^{2,j}\|_{Y'}^2, \quad \forall 1 \leq j \leq k. \end{aligned}$$

Choosing  $\rho_1 \equiv \alpha_a$ ,  $\rho_2 \equiv \varepsilon \alpha_c$  and with the coercivity of  $a(\cdot, \cdot)$  and  $c(\cdot, \cdot)$ , this reduces to

$$\begin{aligned} &\frac{1}{\Delta t} \left( m(e_N^{u,j}, e_N^{u,j}) - m(e_N^{u,j-1}, e_N^{u,j-1}) \right) + a(e_N^{u,j}, e_N^{u,j}) \\ &\quad + \varepsilon c(e_N^{p,j}, e_N^{p,j}) \leq \frac{\|r_N^{1,j}\|_{X'}^2}{\alpha_a} + \frac{\|r_N^{2,j}\|_{Y'}^2}{\varepsilon \alpha_c}, \end{aligned}$$

for all  $1 \leq j \leq k$ . The proposition then follows from applying the sum  $\sum_{j=1}^k$  and  $e_N^{u,0} = \underline{0}$ .

### 3.3 Construction of Reduced Basis Approximation Spaces

Before we discuss how to construct the low-dimensional approximation spaces  $X_N$  and  $Y_N$ , we briefly comment on the computational procedure. RB approximation and error bounds are computed using an Offline–Online strategy: Based on the  $\mu$ -affine expansion of all involved operators, this enables highly efficient computations of the approximations and error bounds. Since much of this machinery is by now standard in RB methods, we do not give further explanations at this point but refer to, e.g., Gerner and Veroy (2011b); Rozza et al. (2008)].

We now turn to the construction of the RB approximation spaces  $X_N$  and  $Y_N$ . The RB method constructs its low-dimensional approximation space  $X_N \times Y_N$  by exploiting the parametric structure of the problem: Basis functions are essentially given by truth solutions associated with several chosen parameter snapshots.

To select these in an efficient way, we here apply the POD greedy procedure first proposed in Haasdonk and Ohlberger (2008): We combine the POD (Proper Orthogonal Decomposition) in  $t^k$ —to capture the causality associated with our evolution equation—with the greedy procedure in  $\mu$  [Binev et al. (2010); Buffa et al. (2009)] enabled through our rigorous and computationally efficient *a posteriori* error bounds. We first greedily select a new parameter  $\mu^*$  for which the RB error bound attains its maximum; we then compute the POD for  $p^{\varepsilon,k}(\mu) - \Pi_{Y_N} p^{\varepsilon,k}(\mu)$  and  $u^{\varepsilon,k}(\mu) - \Pi_{X_N} u^{\varepsilon,k}(\mu)$ ,  $1 \leq k \leq K$ , where  $\Pi_{Y_N}$ ,  $\Pi_{X_N}$  refer to the  $(\cdot, \cdot)_{Y'}$  and  $(\cdot, \cdot)_X$  projections on the current RB spaces  $Y_N$  and  $X_N$ , respectively; finally, we append to our current RB spaces  $Y_N$  and  $X_N$  the respective (first)  $\Delta N$  POD modes. The procedure is repeated until a desired accuracy is satisfied; for more details, we also refer to Knezevic et al. (2011).

For stability reasons [see Gerner and Veroy (2011a); Rozza and Veroy (2007)], the RB approximation space  $X_N$  is

additionally enriched by so-called supremizer functions associated with the chosen pressure basis functions in  $Y_N$ ; we here particularly use Option 2 as defined in Gerner and Veroy (2011b).

## 4. NUMERICAL RESULTS

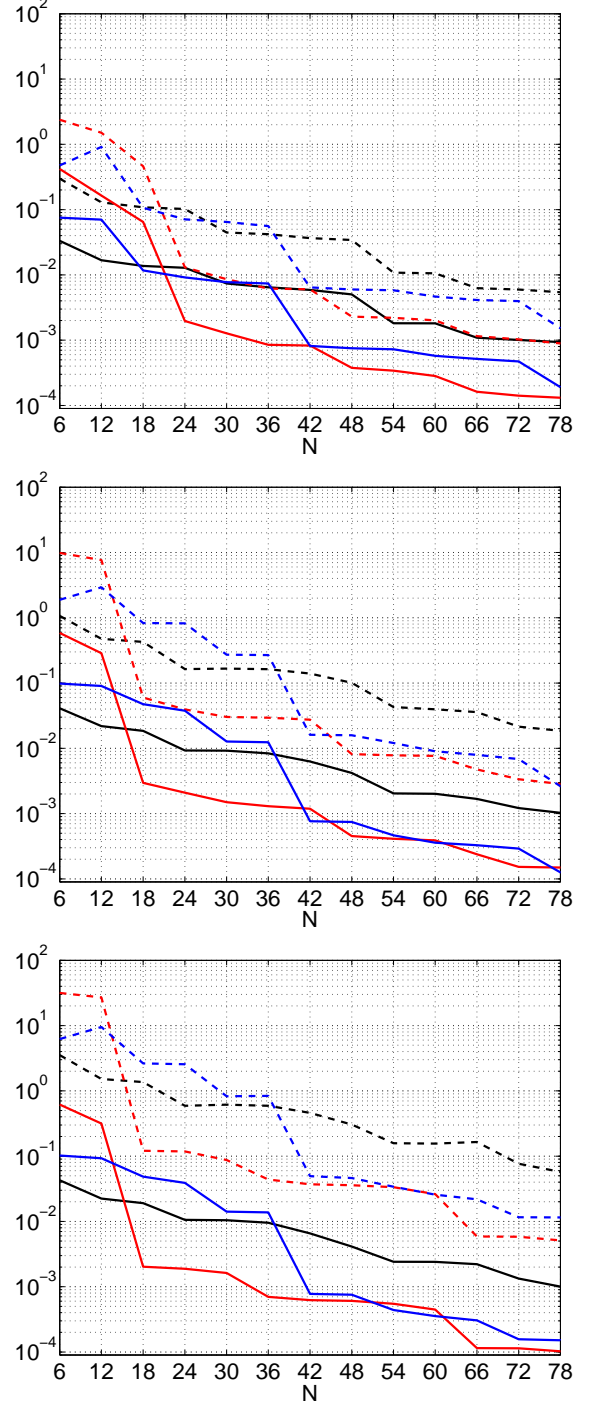


Fig. 1. The errors (24)  $\|e_N^{\varepsilon,K}(\mu)\|_{\mu,\varepsilon}$  (solid lines) and error bounds (30)  $\Delta_N^{\varepsilon,K}(\mu)$  (dashed lines) normalized with respect to  $\|(u^{\varepsilon,K}(\mu), p^{\varepsilon,K}(\mu))\|_{\mu,\varepsilon}$  as functions of  $N$  for  $\varepsilon = 10^{-2}$  (top),  $\varepsilon = 10^{-3}$  (mid),  $\varepsilon = 10^{-4}$  (bottom):  $\mu = (0.1, 0.1)$  (black),  $\mu = (0.3, 0.1)$  (blue),  $\mu = (0.5, 0.5)$  (red).

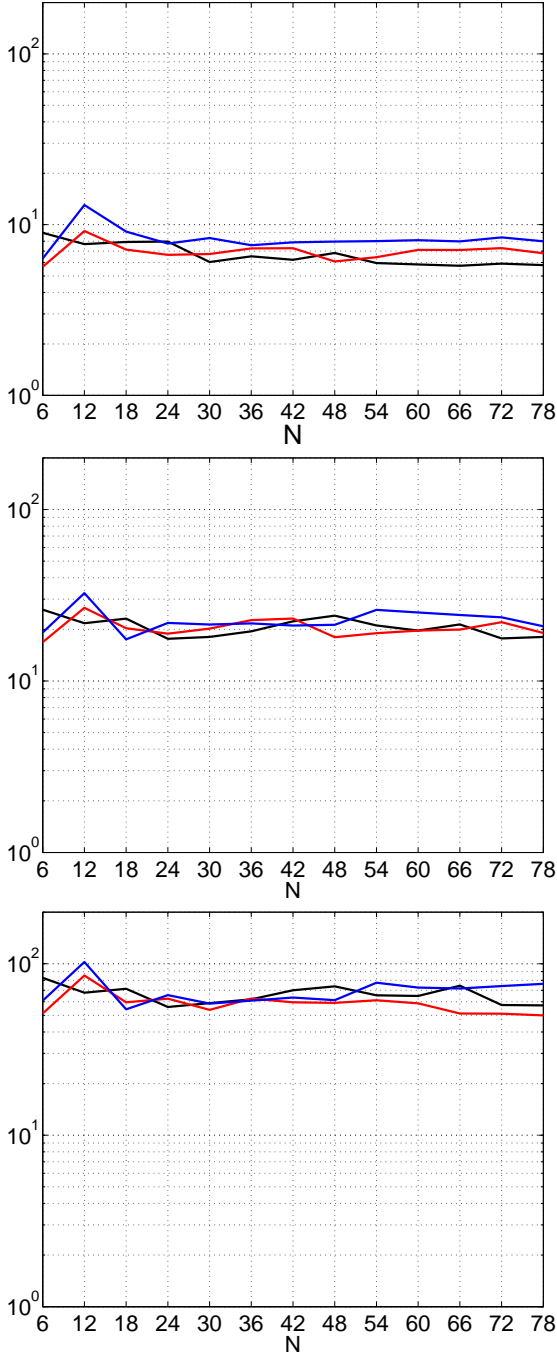


Fig. 2. The effectivities  $\eta_N^{\varepsilon,K}(\mu) \equiv \Delta_N^{\varepsilon,K}(\mu) / \|e_N^{\varepsilon,K}(\mu)\|_{\mu,\varepsilon}$  (see (24), (30)) as functions of  $N$  for  $\varepsilon = 10^{-2}$  (top),  $\varepsilon = 10^{-3}$  (mid),  $\varepsilon = 10^{-4}$  (bottom):  $\mu = (0.1, 0.1)$  (black),  $\mu = (0.3, 0.1)$  (blue),  $\mu = (0.5, 0.5)$  (red).

In this section, we present preliminary numerical results for  $T = 1$  and  $K = 40$  time levels. All computational results are obtained via `rb00mit` [Knezevic and Peterson (2011)], an implementation of the RB framework within in the C++ parallel finite element library `libMesh` [Kirk et al. (2006)]. Lower and upper bounds (27) to the coercivity constants (19), (20) are computed via a successive constraint method (SCM) proposed by Huynh et al. (2007). Using their terminology, we apply it here for  $M_\alpha = \infty$ ,  $M_+ = 0$ , an exhaustive random sample  $\Xi$  of size  $J = 4900$ , and an SCM tolerance of 0.01. The POD greedy procedure

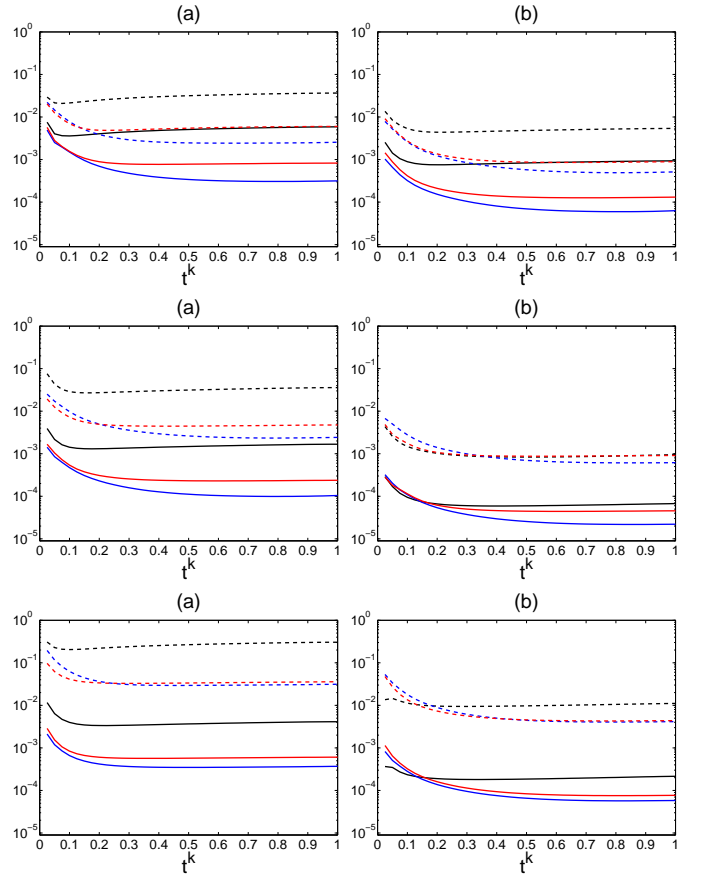


Fig. 3. The errors (24)  $\|e_N^{\varepsilon,k}(\mu)\|_{\mu,\varepsilon}$  (solid lines) and error bounds (30)  $\Delta_N^{\varepsilon,k}(\mu)$  (dashed lines) normalized with respect to  $\|u^{\varepsilon,k}(\mu), p^{\varepsilon,k}(\mu)\|_{\mu,\varepsilon}$  as functions of  $t^k$  for (top line)  $\varepsilon = 10^{-2}$  using (a)  $N = 42$  and (b)  $N = 78$ , (mid line)  $\varepsilon = 10^{-3}$  using (a)  $N = 66$  and (b)  $N = 126$ , and (bottom line)  $\varepsilon = 10^{-4}$  using (a)  $N = 48$  and (b)  $N = 96$ :  $\mu = (0.1, 0.1)$  (black),  $\mu = (0.1, 0.5)$  (blue),  $\mu = (0.5, 0.5)$  (red).

is based on  $\Xi$ ,  $\Delta N = 2$ , and the computationally efficient RB error bound  $\Delta_N^{\varepsilon,K}(\mu)$ .

Fig. 1 now shows the errors  $\|e_N^{\varepsilon,K}(\mu)\|_{\mu,\varepsilon}$  and error bounds  $\Delta_N^{\varepsilon,K}(\mu)$  as functions of the dimension  $N$  for different values of the penalty parameter  $\varepsilon$ ; the associated effectivities  $\eta_N^{\varepsilon,K}(\mu) \equiv \Delta_N^{\varepsilon,K}(\mu) / \|e_N^{\varepsilon,K}(\mu)\|_{\mu,\varepsilon}$  are given in Fig. 2. Fig. 3 presents the errors  $\|e_N^{\varepsilon,k}(\mu)\|_{\mu,\varepsilon}$  and error bounds  $\Delta_N^{\varepsilon,k}(\mu)$  as functions of  $t^k$ . First, we see that the RB error and RB error bounds decrease rapidly as  $N$  increases (see Fig. 1). We obtain stable and rapidly convergent RB approximations and the RB error bounds are meaningful and rigorous. Second, in dependence of the penalty parameter  $\varepsilon$ , the error bounds in particular behave very similarly as in the stationary case [see Gerner and Veroy (2011b)]: For  $\varepsilon = 10^{-2}$ , the error bounds are tight with effectivities essentially less than 10 (see Fig. 2 (top) and also Fig. 3 (top line)); however, as the penalty term is more significant in this case, the associated truth finite element approximation may not be sufficiently accurate. As we decrease  $\varepsilon$  and approach the non-penalized problem (setting  $\varepsilon = 0$ ), the RB error bounds become less sharp (see

Fig. 1 and Fig. 3). The effectivities particularly exhibit a similar  $\sqrt{\varepsilon}$ -dependence on the penalty parameter as derived in Gerner and Veroy (2011b), Corollary 4.1 (see Fig. 2 and also Fig. 3); the effect of the penalty parameter on the effectivities is thus relatively benign, and we obtain useful bounds for reasonably small values of  $\varepsilon$ .

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