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H. Egger

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A class of hybrid mortar finite element methods for interface problems with non-matching meshes*

Herbert Egger†

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Abstract

In this paper, we propose and analyze a family of hybrid finite element methods for interface problems with possibly non-matching meshes. These methods are related to hybrid mixed finite element methods and discontinuous Galerkin methods, as well as Nitsche-type mortaring techniques, but overcome some of the disadvantages of these. By introducing the additional hybrid variable, we obtain methods that allow for subassembling on the subdomain level yielding positive definite global systems. Moreover, the primal unknowns can be eliminated by solving local Dirichlet problems on the subdomains, yielding Schur complement systems for the hybrid variables only. In contrast to dual domain decomposition and mortar methods, the space for the hybrid variable can be chosen with great flexibility without perturbing the stability. We derive the basic a-priori error estimates in energy and L^2-norm, and confirm the theoretical results by numerical tests.

Keywords: interface problems, mortar finite elements, discontinuous Galerkin, hybridization, non-matching grids

AMS subject classification: 65N30, 65N55

1 Introduction

The numerical simulation of practical problems might be challenging due to various reasons, e.g., problems are typically large scale, they involve the coupling of multiple physical models and scales, and the handling of complicated geometies and time-varying interfaces is computationally expensive. Domain decomposition methods [26] provide an attractive framework for approaching such problems, as they allow for splitting of the complicated, large problems into several smaller or simpler subproblems for which adequate solvers might be readily available. An important aspect in domain decomposition is that subproblems can be solved independently of each other to some extent, allowing for a high level of parallelism. The crucial point is of course, to couple the subproblems appropriately, such that they provide a solution for the original problem in the end. To fix ideas and for ease of presentation, let us consider the Dirichlet problem for the Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

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† Center for Computational Engineering Science, RWTH Aachen University, Germany

where Ω is some domain in $\mathbb{R}^d$, $d = 2, 3$ and $f \in L^2(\Omega)$. We suppose that Ω is partitioned into two non-overlapping subdomains Ω_i such that

$$\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}, \qquad \Omega_1 \cap \Omega_2 = \emptyset, \quad \Gamma := \partial\Omega_1 \cap \partial\Omega_2,$$

and assume that the boundaries of the subdomains are sufficiently smooth. The Poisson problem (1) is then equivalent to the following transmission problem, cf. [26]

$$-\Delta u_i = f_i \quad \text{in } \Omega_i, \qquad u_i = 0 \quad \text{on } \partial\Omega \tag{2}$$

such that

$$u_1 - u_2 = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2} = 0 \qquad \text{on } \Gamma. \tag{3}$$

Here and below, $v_i := v|_{\Omega_i}$ denotes the restriction of a function v to Ω_i, and n_i denotes the unit normal vector pointing to the outside of domain Ω_i. We then look for a solution of (2)–(3) in the space

$$\mathcal{V}_0 = \{u \in L^2(\Omega) : u|_{\Omega_i} \in H^1(\Omega_i), \quad u = 0 \quad \text{on } \partial\Omega\}.$$

The interface conditions (3) ensure that a solution u as well as its normal flux $\frac{\partial u}{\partial n}$ are continuous across the interface. From a variational point of view, only one of the conditions has to be satisfied, e.g., ir we assume that $u \in \mathcal{V}_0$ satisfies (2) and $u_1 - u_2 = 0$, then $u \in H^1_0(\Omega)$ and the second condition is satisfied automatically, since according to (2), $\nabla u \in H(\mathrm{div}, \Omega)$ for $f \in L^2(\Omega)$.

A standard approach to incorporate the constraint $u_1 - u_2 = 0$ in a variational framework is via Lagrange multipliers. To be more precise, for $u \in \mathcal{V}_0$ we have $u_1 - u_2 \in H^{1/2}_{00}(\Gamma)$ and a weak formulation of (2) subject to the constraint $u_1 = u_2$ at the interface Γ reads:

Find $u \in \mathcal{V}_0$ and $\lambda \in (H^{1/2}_{00}(\Gamma))'$ such that

$$\begin{aligned} \sum_i \int_{\Omega_i} \nabla u_i \nabla v_i \, dx + \int_\Gamma \lambda(v_1 - v_2) \, ds &= \int_\Omega fv \, dx && \text{for all } v \in \mathcal{V}_0 \\ \int_\Gamma (u_1 - u_2)\mu \, ds &= 0 && \text{for all } \mu \in (H^{1/2}_{00}(\Gamma))'. \end{aligned}$$

For the definition of trace spaces $H^{1/2}_{00}$ and its dual $(H^{1/2}_{00})'$, we refer to [23]. This kind of *dual* approach is the probably most widely used strategy to incorporate interface conditions, with applications in dual domain decomposition and mortar methods. From the equivalence with problem (1) it is clear that $\lambda = \frac{\partial u}{\partial n}|_\Gamma$. In order to obtain an inf-sup stable discretization for the dual domain decomposition method, special care has to be taken in the choice of the space for the Lagrange multiplier [11, 10, 13, 27]. It is however possible to circumvent these difficulties by appropriate stabilization techniques, see [6, 7, 16, 25].

A second possibility to introduce the coupling conditions is offered by *hybrid mixed* methods, which were originally used to facilitate the implementation of $H(\mathrm{div})$ conforming discretizations for dual mixed finite element formualtions, cf. [3, 15, 17]. Here, continuity of the normal flux is enforced via Lagrange multipliers, which for the solution of continuous problem have the meaning of the trace of the primal variable, i.e. $\lambda = u|_\Gamma$. Since inf-sup stable discretizations for the hybrid mixed method are readily available, stability of these methods can be achieved easily by an appropriate choice of spaces, at least on conforming meshes. Now local saddlepoint problems have to be solved now on the subdomains, which clearly limits the practicability of this approach. In case of non-conforming meshes, additional stabilization terms have to be added in order to guarantee an inf-sup condition, cf. [22]. We will present hybrid mixed methods in more detail, and clarify their relation to the proposed hybrid methods in Section 4 below.

By satisfying the interface conditions only approximately, it is possible to formulate methods that do not contain any Lagrange multiplier, and thus circumvent the difficulties due to the sadlepoint structure of the constrained variational problems. Examples of such methods are Nitsche-type methods [5, 8, 9, 24], and other discontinuous Galerkin methods, see [4] and the references therein. Since these methods only involve the primal variables, we will call them *primal* in the sequel. One of the drawbacks of these primal methods is that a lot of coupling is introduced across the interface, such that the the local subproblems cannot be solved independently of of each other, and the global linear systems are less sparse than those arising in standard finite element discretizations. We will recall the definition of a Nitsche-type mortar method in the next section, and show that the unwanted coupling terms can be eliminated.

The hybrid methods that we are going to analyze here are somehere in between the Nitsche and the hybrid mixed formulation: (a) they contain additional (hybrid) variables with a similar meaning as the Lagrange multipliers for the hybrid mixed method, but (b) they do not involve any (dual) flux variable and the systems corresponding to local subproblems are positive definite. Similar to the Nitsche method, stability is achieved via penalization of the jump of the primal variable across the interface. However, due to the introduction of the hybrid variable, the coupling can be reduced to a minimum, i.e., subdomains are only coupled to the interface, and not directly to other subdomains. A feature, that distinguishes the hybrid methods from dual domain decomposition methods, is that the local subproblems are Dirichlet problems, so they are always uniquely solvable. Moreover, no inf-sup condition has to be satisfied, so that the space for the hybrid variable can be chosen with great flexibility. We would like to mention, that the hybridization of a large class of Galerkin finite element methods was discussed already in [17], but an analysis of the resulting methods is missing in this reference. In fact, for certain choices of finite element spaces, our method will coincide with certain variants of methods proposed in [17].

The outline of this article is as follows: In Section 2 we introduce some notation and present the general form of the hybrid methods, which are then analyzed in Section 3. The relation to other methods is discussed in detail in Section 4. The theoretical considerations are confirmed by numerical results in Section 5, and we close with a short discussion.

2 Formulation of the hybrid method

Before we present a class of hybrid finite element methods for problem (2)–(3), let us introduce some notation and basic assumptions. For simplicity, we assume that Ω and Ω_i are bounded polyhedral domains. Let $\mathcal{T}_h(\Omega_i)$ denote conforming triangulations of the subdomains and define by $\mathcal{T}_h := \mathcal{T}_h(\Omega_1) \cup \mathcal{T}_h(\Omega_2)$ the global mesh, which may be non conforming across the interface. For ease of notation, we assume in the sequel that the triangulations are simplicial and quasi-uniform with meshsize h, but our results can easily be generalized to shape regular triangulations and more general elements.

We employ here the standard notation used in the context of discontinuous Galerkin methods, cf. [4]. For piecewise smooth functions $u \in \mathcal{V} := \{u \in L^2(\Omega : u_i = u|_{\Omega_i} \in H^1(\Omega))\}$, the jump respectively the mean value at the interface Γ are defined by

$$[u] := u_1 n_1 + u_2 n_2 \qquad \text{and} \qquad \{u\} := \frac{1}{2}(u_1 + u_2),$$

where n_i denotes outwards directed unit normal vector on $\partial\Omega_i$. The mean value for vector valued functions is defined component wise. For brevity, we further use the following notation

$$(u, v)_\Omega := \int_\Omega uv\, dx \qquad \text{and} \qquad \langle u, v\rangle_\Gamma := \int_\Gamma uv\, ds,$$

and we denote the corresponding norms by $\|u\|_\Omega := (u, u)_\Omega^{1/2}$ and $|u|_\Gamma := \langle u, u\rangle_\Gamma^{1/2}$.

For motivation of the hybrid methods, and in order to clarify their relation to discontinuous Galerkin and Nitsche methods, we shortly recall the definition of the Nitsche-mortar method presented in [9]; see also [2, 4] for related methods in the framework discontinuous Galerkin methods. Let $\mathcal{V}_h(k)$ denote the composition of standard finite element spaces of order k on the subdomains, i.e.,

$$\mathcal{V}_h(k) := \{v_h \in \mathcal{V}_0 : v_h|_T \in P^k(T) \text{ for } T \in \mathcal{T}_h\}. \tag{4}$$

The polynomial degree is assumed to be constant here for simplicity, but the results naturally apply to discretizations using variable polynomial degree. If there is no need to refer to the polynomial degree explicitly, we also write $\mathcal{V}_h$ instead of $\mathcal{V}_h(k)$. The Nitsche-mortar method is then defined by the following variational problem [9].

Find $u_h \in \mathcal{V}_h$ such that

$$\sum_i (\nabla u_h, \nabla v_h)_{\Omega_i} - \langle\{\nabla u_h\}, [v_h]\rangle_\Gamma - \langle[u_h], \{\nabla v_h\}\rangle_\Gamma + \tfrac{\alpha}{h}\langle[u_h], [v_h]\rangle_\Gamma = (f, v_h)_\Omega$$

holds for all $v_h \in \mathcal{V}_h$.

This method is consistent, and also stable with respect to an appropriate energy norm if the stabilization parameter α is chosen sufficiently large. The third term in the bilinear form is introduced to retain symmetry, and the fourth term penalizes the jump of u_h. The second to fourth term of the bilinear form introduce some coupling in the corresponding linear systems, which inhibits the easy solution of the local subproblems. We will show now, how this drawback can be eliminated by hybridization.

Let us formally introduce the mean value of u as a new variable λ. In this way, we obtain new ways to express the jump of u, namely

$$[u] = u_1 n_1 + u_2 n_2 = 2(u_1 - \lambda) n_1 = 2(u_2 - \lambda) n_2,$$

which we can use to reduce the coupling in the Nitsche-type method. For the formulation of the numerical method, we will choose some function space

$$\mathcal{M}_h \subset L^2(\Gamma)$$

for discretization of the hybrid variable. Here, the subscript h shall indicate that $\mathcal{M}_h$ is a space that is used in the numerical method, but as we will see in Section 3 below, the space $\mathcal{M}_h$ may not be related to the meshsize at all. From an analytical point of view, we may even choose $\mathcal{M}_h = L^2(\Gamma)$ or $\mathcal{M}_h = \mathrm{span}\{1\}$. Substituting λ into the Nitsche method and after some simple manipulations, we arrive at the following method.

Method 1 (Hybrid mortar method)**.** *Find $(u_h, \lambda_h) \in \mathcal{V}_h \times \mathcal{M}_h$ such that*

$$B(u_h, \lambda_h; v_h, \mu_h) = F(v_h) \qquad \text{for all} \quad (v_h, \mu_h) \in \mathcal{V}_h \times \mathcal{M}_h,$$

where

$$\begin{aligned} B(u_h, \lambda_h; v_h, \mu_h) := \sum_i (\nabla u_h, \nabla v_h)_{\Omega_i} &- \langle \tfrac{\partial u_h}{\partial n}, v_h - \mu_h \rangle_{\partial\Omega_i \cap \Gamma} \\ &- \langle u_h - \lambda_h, \tfrac{\partial v_h}{\partial n} \rangle_{\partial\Omega_i \cap \Gamma} + \tfrac{2\alpha}{h} \langle u_h - \lambda_h, v_h - \mu_h \rangle_{\partial\Omega_i \cap \Gamma} \end{aligned}$$

and

$$F(v_h, \mu_h) := (f, v_h)_\Omega.$$

Remark 1. Method 1 actually defines a class of methods depending on the special choice for the space $\mathcal{M}_h$. As we will show below, we have a great flexibility in choosing this space, which enters the analysis only through its approximation properties. We will present some possible choices and discuss generalizations of our results and relations to other methods in Section 4. Although we formally derived our method by introducing λ as the mean value of u and inserting the jump relations into the Nitsche-mortar method, the hybrid Method 1 is not equivalent to the former, i.e. we obtain different solutions u_h in general. In particular, the finite element solution (u_h, λ_h) of Method 1 will typically not satisfy the condition $\lambda_h = \{u_h\}$. If $\mathcal{M}_h$ is sufficiently rich, we have instead

$$\lambda_h = \{u_h\} - \tfrac{h}{2\alpha} [\nabla u_h],$$

where $[\nabla u_h] = \nabla u_1 n_1 + \nabla u_2 n_2$ denotes the jump of the normal flux.

Remark 2. In the derivation of our method we added the third term in order to ensure symmetry of the resulting bilinear form. In a similar manner as for interior penalty methods [2, 4], we could also omit this term or add it with the opposite sign, i.e., we could replace the third term of the bilinear form by $-\beta \langle u_h - \lambda_h, \frac{\partial v_h}{\partial n} \rangle_{\partial\Omega_i \cap \Gamma}$ with $\beta \in \mathbb{R}$. From a computational point of view, the choices $\beta \in \{1, 0, -1\}$, are most interesting, as they allow to obtain symmetry, introduce least coupling, or provide higher stability than other choices. We will present our results only for the symmetric case below, and comment on the other cases in remarks.

An important property of our method, which holds regardless of the special choice for the space $\mathcal{M}_h$ is the following.

Proposition 3 (Consistency)**.** *Let $u \in H^{3/2+\epsilon}(\Omega)$ denote the solution of* (2) *and let (u_h, λ_h) be the finite element solution of Method 1. Then*

$$B(u - u_h, u - \lambda_h; v_h, \mu_h) = 0 \qquad \text{for all } (v_h, \mu_h) \in \mathcal{V}_h \times \mathcal{M}_h.$$

Proof. Let u denote the solution of (2) respectively (1), and let $v_h \in \mathcal{V}_h$. Then by our regularity assumption $\frac{\partial u}{\partial n}$ is single valued at the interface Γ, and we obtain

$$(f, v_h) = \sum_i (\nabla u, \nabla v_h) - \langle \tfrac{\partial u}{\partial n}, v_h \rangle_{\partial\Omega_i \cap \Gamma} = \sum_i (\nabla u, \nabla v_h) - \langle \tfrac{\partial u}{\partial n}, v_h - \mu_h \rangle_{\partial\Omega_i \cap \Gamma}.$$

The remaining terms of the bilinear form are zero, since $u \in H^1(\Omega)$ implies $[u] = 0$. $\square$

Remark 4. In the previous proposition, we had to assume that $u \in H^{3/2+\epsilon}(\Omega)$ such that the normal flux $\frac{\partial u}{\partial n} \in L^2(\Gamma)$ is well-defined. This is needed only, if we want to allow for general multiplier functions $\mu_h \in L^2(\Gamma)$. If we choose the multiplier space such that $\mathcal{M}_h \subset H^{1/2}_{00}(\Gamma)$, this additional smoothness assumption can be dropped. The assumption $u \in H^{3/2+\epsilon}(\Omega)$ however easily translates to conditions on the domain and the data, and is used as a standing assumption in domain decomposition and discontinuous Galerkin literature. So in order to avoid technical difficulties and to state relatively general results, we use that assumption at several places below.

3 Analysis of the hybrid method

Let us start by introducing the natural norms for the analysis of Method 1, see, e.g. [9, 13]. For $s \geq 0$ and piecewise smooth functions functions $v \in H^s(\Omega_1 \cup \Omega_2)$, we define the broken Sobolev norm by

$$\|v\|^2_{s,h} := \sum_i \|v\|^2_{H^s(\Omega_i)}.$$

For ease of notation, we further define the norm of a function to take the value $+\infty$ if the function does not satisfy the regularity requirements. In case $s > 1/2$, we can further define the discrete trace norms

$$|v|^2_{1/2,h} := \sum_i \tfrac{1}{h}|v|^2_{\partial\Omega_i \cap \Gamma} \qquad \text{and} \qquad |v|^2_{-1/2,h} := \sum_i h|v|^2_{\partial\Omega_i \cap \Gamma}.$$

This definition naturally extends to functions in $L^2(\Gamma)$. Similar as in the continuous case, the following (duality) inequality holds for the discrete norms

$$\sum_i \langle u, v\rangle_{\partial\Omega_i \cap \Gamma} \leq |u|_{1/2,h}|v|_{-1/2,h}.$$

Stability of the hybrid method will be proven with respect to the following mesh dependent energy norm. For functions $v \in \mathcal{V}_0$ and $\mu \in L^2(\Gamma)$ we define

$$|||(v, \mu)|||_{1,h} := (\sum_i \|\nabla v\|^2_{\Omega_i} + |v - \mu|^2_{1/2,h})^{1/2},$$

and for proving the boundedness, we will further utilize the slightly stronger norm

$$|||(v, \mu)|||_{1,*} := (|||(v, \mu)\ |||^2_{1,h} + |\tfrac{\partial v}{\partial n}|^2_{-1/2,h})^{1/2},$$

where we require that v is piecewise smooth such that $\frac{\partial v}{\partial n}$ is well defined in $L^2(\partial\Omega_i \cap \Gamma)$. This is obviously the case for the finite element functions. In fact, the two norms are equivalent for function $v \in \mathcal{V}_h(k)$, $\lambda \in \mathcal{M}_h$, which follows readily from discrete trace inequalities and the standard scaling arguments.

Let us first prove stability, for which we require no further restriction on the space $\mathcal{M}_h$ for the hybrid variable.

Proposition 5 (Stability)**.** *Let B be defined as in Method 1 with $\alpha > 0$ sufficiently large. Then the ellipticity estimate*

$$B(u_h, \lambda_h; u_h, \lambda_h) \geq \tfrac{1}{2} \, ||| \, (u_h, \lambda_h) |||_{1,h}^2$$

holds for all $(u_h, \lambda_h) \in \mathcal{V}_h \times \mathcal{M}_h$. The lower bound for α is independent of h, but only depends on the shape of the triangles attached to the interface.

Proof. Inserting $v_h = u_h$ and $\mu_h = \lambda_h$ we obtain

$$B(u_h, \lambda_h; u_h, \lambda_h) = \|\nabla u_h\|_{0,h}^2 + \alpha |u_h - \lambda_h|_{1/2,h}^2 - 2 |\tfrac{\partial u_h}{\partial n}|_{-1/2,h} |u_h - \lambda_h|_{1/2,h}.$$

A standard scaling argument shows that $|\frac{\partial u_h}{\partial n}|_{-1/2,h} \leq C_I \|\nabla u_h\|_{0,h}$, and the result now follows by choosing $\alpha > C_I^{-1}$ and Youngs inequality. □

The result shows that the stability is not influenced at all by the choice of $\mathcal{M}_h$. In fact, the stability estimate is formally valid even for the trivial case $\mathcal{M}_h = \{0\}$. The same observation holds true for the boundedness of the bilinear form.

Proposition 6 (Boundedness)**.** *Let the conditions of Proposition 5 be valid. Then the estimate*

$$B(u - u_h, \lambda - \lambda_h; v_h, \mu_h) \leq C \, ||| \, (u - u_h, \lambda - \lambda_h) \, |||_{1,*} \, |||(v_h, \mu_h)|||_{1,h}$$

holds for all (u_h, λ_h), $(v_h, \mu_h) \in \mathcal{V}_h \times \mathcal{M}_h$, and for all $u \in \mathcal{V}_0$ with $u|_{\Omega_i} \in H^{3/2+\epsilon}(\Omega_i)$ and $\lambda \in L^2(\Gamma)$. The constant C is independent of the meshsize h.

The special choice of $\mathcal{M}_h$ does not influence the ellipticity and boundedness of the bilinear form, but it determines the approximation properties with respect to the mesh dependent norms. To obtain order optimal error estimates, we require that $\mathcal{M}_h$ is sufficiently rich. For brevity, we analyse a special choice in detail in the sequel, and turn to generalizations at the end of this section.

Let $\mathcal{G}_h$ denote a quasi-uniform triangulation of the interface Γ with meshsize h comparable to that of the triangulation $\mathcal{T}_h$, and define

$$\mathcal{M}_h(k) := \{\mu_h \in L^2(\Omega) : v|_S \in P_k(G) \quad \text{for all} \quad G \in \mathcal{G}_h\}. \tag{5}$$

Quasi uniformity of the mesh and comparable mesh sizes are only needed to simplify the presentation; the results however easily generalize to shape regular meshes and arbitrary mesh sizes. We are now in the position to characterize the approximation properties of the finite element spaces $\mathcal{V}_h(k)$ and $\mathcal{M}_h(k)$. For piecewise smooth functions v with $v|_{\Omega_i} \in H^{3/2+\epsilon}(\Omega_i)$, let us define the interpolant $v_h := \mathcal{I}_k v \in \mathcal{V}_h$ subdomain-wise by $v_h|_{\Omega_i} = \mathcal{I}_k v|_{\Omega_i}$, where $\mathcal{I}_k$ denotes an H^1 interpolation operator on the subdomains, cf. e.g. [14]. For interface functions, we utilize the L^2 orthogonal projector and define the interpolant by $\mu_h := \Pi_k \mu$ for functions $\mu \in L^2(\Gamma)$. With these definitions, the following interpolation error estimates follow readily by the usual scaling arguments.

Proposition 7. *Let $u \in H^1(\Omega)$ be such that $u|_{\Omega_i} \in H^{s+1}(\Omega_i)$ for some $1/2 < s \leq k$. Then*

$$|||(u - \mathcal{I}_k u, u - \Pi_k u) \, |||_{1,*} + ||| \, (u - \mathcal{I}_k u, u - \Pi_k u)|||_{1,h} \leq C h^s \|u\|_{s+1,h}.$$

The a-priori error estimate in the energy norm now follows from the ellipticity, boundedness and consistency of the bilinear form using standard arguments.

Theorem 8 (Energy norm estimate)**.** *Let (u_h, λ_h) denote the solution of Method 1 with $\mathcal{M}_h = \mathcal{M}_h(k)$ and α sufficiently large. Moreover, let u be the solution of* (2)*. Then the a-priori error estimate*

$$|||(u - u_h, u - \lambda_h)|||_{1,h} \leq Ch^s \|u\|_{s+1,h}$$

holds for $1/2 < s \leq k$ with constant C independent of the mesh size h.

Proof. The results follows directly from PRoposition 5 (stability), consistency of the method, Proposition 6 (boundedness), and the interpolation error estimates of Proposition 7. □

Remark 9. The same estimates also hold for the nonsymmetric variants of the hybrid mortar method, see Remark 2. For $\beta = -1$, there is no restriction on the size of α, i.e., the choice $\alpha = 1$ is possible.

For the symmetric hybrid method we can apply the Aubin-Nitsche trick to obtain the optimal L^2-norm estimate as well.

Theorem 10 (L^2-norm estimate)**.** *Let the assumption of Theorem 8 be valid, and let the problem* (1) *be H^2-regular. Then there exists a constant C independent of the mesh size h such that*

$$\|u - u_h\| \leq Ch^{s+1} \|u\|_{s+1,h}$$

holds for $1/2 < s \leq k$.

Proof. Let w denote the solution of (1) with f replaced by $u - u_h$. Then

$$\|u - u_h\|^2 = B(u - u_h, u - \lambda_h; w, w) = B(u - u_h, u - \lambda_h; w - \mathcal{I}_k w, w - \Pi_k w),$$

where we used symmetry and consistency of the bilinear form. Similar as in Proposition 6 we obtain

$$B(u, \lambda; v, \mu) \leq |||(u, \lambda)\,|||_{1,*}\, |||(v, \mu)|||_{1,*},$$

for all piecewise H^s functions u, v with $s > 3/2$, and for all λ, $\mu \in L^2(\Omega)$. Since we assumed H^2-regularity of our problem, we obtain $\|w\|_{2,h} \leq c\, \|u - u_h\|$, and the result follows readily by applying the interpolation error estimates of Proposition 7 and Theorem 8. □

Remark 11. Improved L^2 error estimates also hold for problems that are only H^r-regular for some $1 < r \leq 2$, in which case we have $\|u - u_h\| \leq Ch^{s+r}\|u\|_{s,h}$; see [21] for similar results. The methods with $\beta \neq 1$, see Remark 2, are not consistent for the adjoint problem $-\Delta w = u - u_h$, and do not yield optimal L^2 errors in general; cf. [4] for the definition of *adjoint consistency* and a related discussion in the framework of discontinuous Galerkin methods.

At the end of this section, let us discuss some generalizations of our results. In view of the proofs of Theorem 8 and 10, the space $\mathcal{M}_h$ only has to satisfy some approximation property, while the stability and boundedness are not influenced. Let us assume that the hybrid variable space approximates functions on the interface sufficiently well, i.e.

$$\inf_{\mu_h \in \mathcal{M}_h} |\lambda - \mu_h| \leq Ch^r |\lambda|_{r,\Gamma} \qquad \text{for all } \lambda \in H^r(\Gamma) \quad \text{and some } r > 0. \tag{6}$$

The following error estimate then follows similarly as the previous ones.

Theorem 12. *Let $u \in H^{3/2+\epsilon}(\Omega)$ be the solution of* (1) *and let (u_h, v_h) be the solution of Method 1 with $\mathcal{M}_h$ satisfying* (6) *for $r \geq s + 1/2$. Then the error estimates of Theorem 8 and 10 hold.*

Proof. We only have to show that an approximation estimate similar to that of Proposition 7 holds: For $u \in H^1(\Omega)$ we have $\lambda = u|_\Gamma \in L^2(\Gamma)$, and by (6) and the trace inequality we obtain

$$\inf_{\mu_h \in \mathcal{M}_h} |u - \mu_h|^2_{1/2,h} \leq C h^{2s} \|u\|^2_{s+1,h}.$$

Instead of the explicit definition of the interpolant $\Pi_k u$ at the interface, we can now just work with any sufficiently well approximating element. This yields the necessary approximation error estimates in the norms $|||\cdot|||_{1,h}$ and $|||\cdot|||_{1,*}$ and the result follows as in the proof of Theorem 8 and 10. $\square$

Remark 13. Theorem 12 allows to choose $\mathcal{M}_h$ completely independent of the mesh, i.e., if the interface is sufficiently smooth, global function can be used for the hybrid variable. If Ω_i is moving, or if the mesh is refined or coarsened during the computation, such a choice alleviates the handling of the interface terms. A similar approach was considered in [20] for a stabilized dual domain decomposition method. Alternatively, one can employ $\mathcal{M}_h \subset H^{1/2}_{00}(\Gamma)$, e.g., by using the trace of function in $\mathcal{V}_h$ on one of the subdomains. In this case, the additional regularity assumption $u \in H^{3/2+\epsilon}(\Omega)$, which was needed to make all interface terms appearing in the bilinear form well-defined can be omitted in the consistency statement.

Remark 14. It would be possible in principle, to decompose the domain completely into single elements. The proposed method can then be viewed as a hybrid version of discontinuous Galerkin methods, and our results apply also to that case. In fact, for a special choice of $\mathcal{M}_h$, Method 1 coincides with one of the hybridized discontinuous Galerkin methods presented in [17], where the possibility to couple different discretization methods, and the potential application to nonconforming discretizations has been pointed out, while an analysis of the proposed methods is missing. In the traditional hybrid mixed formulation, $H(\mathrm{div})$ conforming elements are used for discretization of the flux, and the jump penalization term can be omitted; see also [18] for application to convection-diffusion problems and a similar analysis as above, and the remarks in the following section.

4 Some comments on the relation to mixed finite elements and other mortar methods

As mentioned in the introduction, the proposed hybrid method can be seen somewhere in between the Nitsche-mortar and the hybrid mixed methods. While the connection to the Nitsche-type methods is obvious from the derivation, let us clarify the relation to mixed problems in more detail. The mixed form of (2)–(3) reads

$$\sigma + \nabla u = 0, \quad \mathrm{div}\, \sigma = f \quad \text{in } \Omega_i,$$
$$u_1 - u_2 = 0, \quad \sigma_1 n_1 + \sigma_2 n_2 = 0 \quad \text{on } \Gamma.$$

Enforcing continuity of the flux $\sigma_1 n_1 + \sigma_2 n_2 = 0$ via Lagrange multipliers, yields the following hybrid mixed variational problem [3, 15]: Find $\sigma \in \mathcal{S} := \{\tau \in L^2(\Omega)^d : \tau|_{\Omega_i} \in H(\mathrm{div}, \Omega_i)\}$

and $u \in L^2(\Omega)$ such that

$$\sum_i (\sigma, \tau)_{\Omega_i} - (u, \operatorname{div} \tau)_{\Omega_i} + \langle \lambda, \tau n \rangle_{\partial\Omega_i \cap \Gamma} = 0 \tag{7}$$

$$\sum_i (\operatorname{div} \sigma, v)_{\Omega_i} + \langle \sigma n, \mu_h \rangle_{\partial\Omega_i \cap \Gamma} = -(f, v)_\Omega \tag{8}$$

holds for all $\tau \in \mathcal{S}$, $v \in L^2(\Omega)$ and $\mu_h \in H^{1/2}_{00}(\Gamma)$. A standard discretization for the hybrid mixed problem is to choose $\sigma_h \in \mathcal{S}_h(k)$, $u_h \in \mathcal{W}_h(k)$ and $\lambda_h \in \mathcal{M}_h(k)$ with

$$\begin{aligned} \mathcal{S}_h(k) &:= \{\sigma_h \in L^2(\Omega) : \sigma|_{\Omega_i} \in H(\operatorname{div}; \Omega_i),\ \sigma|_T \in RT_k(T) \text{ for all } T \in \mathcal{T}_h\} \\ \mathcal{W}_h(k) &:= \{w_h \in L^2(\Omega) : w \in P_k(T) \text{ for all } T \in \mathcal{T}_h\}, \end{aligned}$$

and $\mathcal{M}_h(k)$ as above. If the mesh $\mathcal{T}_h$ is conforming, the discrete method is equivalent to the mixed method with conforming spaces.

The introduction of the Lagrange multipliers allows to reduce the overall system by statically eliminating the σ and u unknowns on the subdomains, yielding a global positive definite Schur complement system for λ only; for details we refer to [3, 15]. Once, λ has been determined, the solution $u \in \mathcal{W}_h(k)$ can be reconstructed by solving local saddlepoint problems, which amounts to the solution of the local (primal) subdomain problems in our case. Substituting ∇u for σ in (7) and ∇v for τ in (8), and integrating by parts formally yields the variational form of Method 1, apart from the stabilization terms.

While for conforming meshes, inf-sup stability of the global system (7)–(8) follows readily from the appropriate choice of the finite element spaces, some sort of stabilization has to be used in order to obtain stability also for non-conforming meshes, cf. [22].

We also would like to point our relations to stabilized mortar methods: The construction of of approriate finite element spaces for the Lagrange multiplier (which has the meaning of a normal flux in this situation, i.e., $\lambda = \frac{\partial u}{\partial n}$) is relatively complicated [10, 12, 27], inf-sup stability can be shown for rather general choices, if stabilization is added [16].

5 Numerical tests

For illustration of the theoretical results of the previous sections, we consider the following model problem:

$$-\Delta u = f \quad \in \Omega = (-1, 1) \times (0, 1), \qquad u = 0 \quad \text{on } \partial\Omega, \tag{9}$$

with right hand side $f|_{\Omega_1} = 0$ and $f|_{\Omega_2} = 1$. As approximation for the exact solution, the problem is first solved on a very fine grid by a standard H^1-conforming finite element method.

We consider a partition of the domain into two subdomains $\Omega_1 = (-1, 0) \times (0, 1)$ and $\Omega_2 = \Omega \setminus \Omega_1$. For meshing the subdomains, we use different meshsizes $h_1 = h/3$ and $h_2 = h/2$, so that the global mesh is not conforming across the interface, see Figure 1. The interface mesh $\mathcal{G}_h$ is constructed by a segmentation containing all interface points of both subdomain meshes.

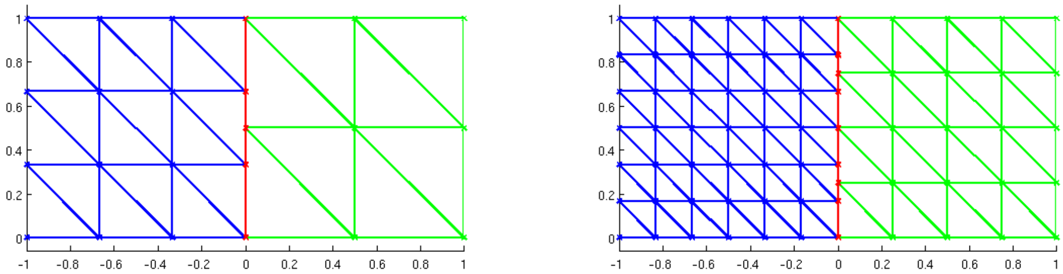

Figure 1: Meshes for $h = 1$ and $h = 1/2$, with meshsizes $h_1 = h/3$ and $h_2 = h/2$ on the individual subdomains.

For the numerical solution of the interface problem corresponding to (9), we employ Method 1 with $\mathcal{V}_h(1)$ and $\mathcal{M}_h(1)$. The global finite element system has the form

$$\begin{bmatrix} A_1 & 0 & B_1^T \\ 0 & A_2 & B_2^T \\ B_1 & B_2 & C \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ G \end{bmatrix}$$

with $C = C_1 + C_2$ and $G = G_1 + G_2$. Note that all terms with index i can be assembled on the corresponding subdomain Ω_i. Since Ω is convex, and $f \in L^2$, standard shift theorems guarantee that $u \in H^2(\Omega)$. We can therefore expect convergence of optimal order, i.e. h in the energy respectively broken H^1 norm, and h^2 in the L^2 norm. The numerical results listed in Table 1, confirm the theretical results of the previous sections.

h	$\|u - u_h\|_{1,h}$	$\|u - u_h\|_0$
0.5	0.032155	0.001989
0.25	0.016644	0.000514
0.125	0.008481	0.000127
0.0625	0.004024	0.000029
rates	0.997	2.032

Table 1: Errors of Method 1 in broken H^1 and L^2 norm. The subdomains are meshed with meshsizes $h_1 = h/3$ and $h_2 = h/2$, cf. Figure 1.

In Figure 2, we display the finite element solutions obtained on a two grids, and we compare the interface values given by the hybrid variable with the traces of the primal functions u_i on the subdomains in Figure 3.

As already indicated above, one of the advantages of the hybrid approach over Nitsche-type mortaring is, that it allows to locally eliminate the primal variables and solve the resulting Schur complement system $S\lambda = G$, with

$$S = A_3 - B_1 A_1^{-1} B_1^T - B_2 A_2^{-1} B_2^T, \qquad G = F_3 - B_1 A_1^{-1} F_1 - B_2 A_2^{-1} F_2,$$

which makes it attractive as domain decomposition technique. In this context, we would like to emphasize, that solving with A_1 and A_2 corresponds to local Dirichlet problems, hence A_1, A_2 are positive definite.

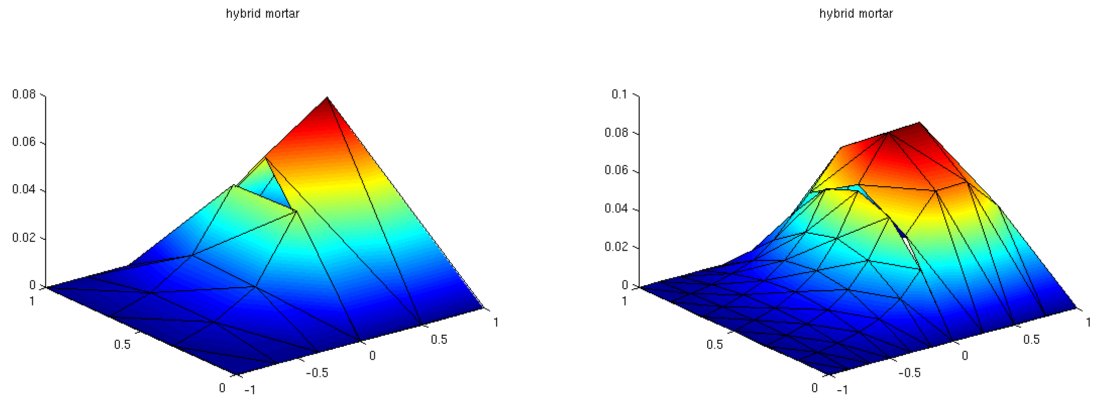

Figure 2: Finite element solutions for Method 1 with $h = 1$ and $h = 1/2$, and meshsizes $h_1 = h/3$ and $h_2 = h/2$ on the subdomains. The black line denotes interface value of the true solution.

The Schur complement matrix essentially amounts to the discretization of a differential operator of order 1, so we expect that the condition number can be reduced from $O(h^{-2})$ for the full system to $O(h^{-1})$ for the Schur complement system, which we found to be in good agreement with our numerical tests, cf. Table 2.

h	0.5	0.25	0.125	0.0625	0.03125	rates
$\kappa(S)$	59.3	121.9	314.3	633.3	1265.2	-1.10
$\kappa(K)$	1163.9	3718.1	13133.3	51810.6	197614.3	-1.82

Table 2: Condition numbers for the full system K and the Schur complement system S of Method 1.

6 Conclusion and further directions

In this paper, we proposed and analyzed a class of hybrid finite element methods for interface problems on possibly non-matching meshes. As opposed to dual domain decomposition and related mortar methods, the hybrid approach is very flexible with respect to discretization, i.e., stability is not affected by the special choice of the space $\mathcal{M}_h$ for the hybrid variable. In contrast to Nitsche-type methods, the coupling of the subproblems occurs only via the interface functions, which decreases the coupling between the subdomains to a minimum and enhances parallel solution techniques. Additionally, it is possible, to eliminate all primal unknowns already on the subdomain level by solving local Dirichlet problems, yielding a positive definite Schur complement system for the hybrid variables only.

For presenting the general framework, we restricted ourselves here to the Dirichlet problem for the Poisson equation and only two subdomains. The generalization to many subdomains is straight forward, and the applications to more involved problems like fluid-structure interaction [1, 19] and problems with moving domains will be topics of future research.

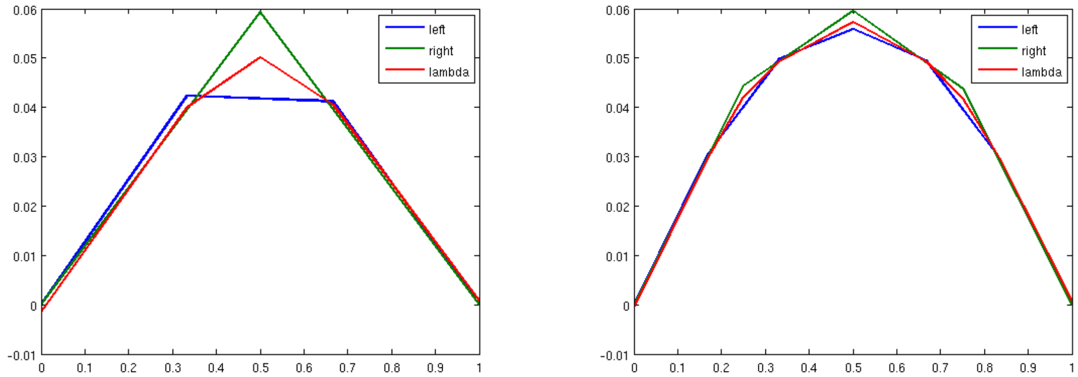

Figure 3: Hybrid variable λ_h of Method 1 (red) and traces of the primal variables u_i (blue, green) for different discretization levels $h = 1$ and $h = 1/2$, and meshsizes $h_1 = h/3$ and $h_2/2$ on the subdomains

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