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A HYBRID MORTAR FINITE ELEMENT METHOD FOR THE STOKES PROBLEM

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ABSTRACT. In this paper, we consider the discretization of the Stokes problem on domain partitions with non-matching meshes. We propose a hybrid mortar method, which is motivated by a variational characterization of solutions of the corresponding interface problem. For the discretization of the subdomain problems, we utilize standard inf-sup stable finite element pairs. The introduction of additional unkowns at the interface allows to reduce the coupling between the subdomain problems, which comes from the variational incorporation of interface conditions. We present a detailed analysis of the hybrid mortar method, in particular, the discrete inf-sup stability condition is proven under weak assumptions on the interface mesh, and optimal a-priori error estimates are derived with respect to the energy and L^2-norm. For illustration of the results, we present some numerical tests.

Keywords: Stokes equation, interface problems, discontinuous Galerkin methods, hybridization, mortar methods, non-matching grids

AMS subject classification: 65N30, 65N55

1. Introduction

Several interesting applications in computational fluid dynamics involve moving geometries, multiple physical phenomena, or discontinuous material properties. As typical examples, let us mention the flow around spinning propellers, fluid-structure interaction, groundwater contaminant transport, or multiphase flows.

For problems with varying geometries or interfaces, it may be convenient to use discretizations that are not matching at the interfaces; e.g., in the hydrodynamic simulation of rotating propellers it is common practice to generate independent meshes for the rotor and the stator domain. Continuity of the solution is then obtained by imposing appropriate coupling conditions on the cylindrical interface.

Methods that incorporate interface conditions in a variational framework allow to deal with non-matching meshes more or less automatically. A prominent example are classical mortar methods [5], which enforce jump conditions across the interface by Lagrange multipliers. While these methods are well-studied, cf. e.g. [7, 26], they also have certain peculiarities, e.g., the space of Lagrange multipliers has to be chosen with care to obtain stability; additionally, the resulting linear systems are saddle point problems and require appropriate solvers.

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An alternative variational approach for the discretization of interface problems is offered by Nitsche-type mortaring [3]. Such discretizations avoid the use of Lagrange multipliers, and consequently, the resulting linear systems are positive definite and can be solved with standard iterative methods, e.g., the method of conjugate gradients.

One drawback of these mortar methods of Nitsche-type is that a lot of (unnatural) coupling is introduced across the interface. This complicates the independent solution of the subdomain problems, and clearly limits the applicability in domain decomposition algorithms [24].

The strong coupling of the subdomain problems can however be relaxed by hybridization [10], i.e., by the introduction of additional unknowns at the interface. The hybrid methods yield again positive definite linear systems, and inherit the great flexibility in the choice of ansatz spaces from the Nitsche-type methods, but without introducing their strong coupling.

The aim of this paper is to extend the theoretical framework of hybrid mortar methods [10, 11] to the Stokes system. We derive a variational characterization for the Stokes interface problem, which, via Galerkin discretization, naturally gives rise to a hybrid mortar method. We first study in detail a two dimensional model problem, and then discuss how the analysis can be generalized in order to cover a variety of finite element discretizations in two or three dimensions. Stability of the discrete problems is obtained under mild conditions on the domain partition; in particular, the meshes on the subdomains can be chosen completely independent from each other.

Some further related work, we would like to mention is the following: The discretization of Stokes interface problems was investigated in the context of classical mortar methods for instance in [4, 12], and in the framework of discontinuous Galerkin methods in [16, 25, 13]. Hybridization has also been used for the formulation of discontinuous Galerkin methods for Stokes flow [22], and the analysis of the vorticity formulation of Stokes' problem [9]. The approach discussed in this paper however differs in the type of application or discretization.

The plan for our presentation is as follows: In Section 2, we state the Stokes interface problem and derive a variational characterization of solutions to this problem. This characterization is the starting point for the formulation of a hybrid mortar finite element method, and in Section 3, we present in detail the stability and error analysis for a specific discretization of a two dimensional model problem. Section 4 then discusses the generalization of the results to three dimensions and more general inf-sup stable finite element pairs. We also relax the conditions on the domain partition, and make remarks concerning further generalizations. Section 5 presents some numerical tests in support of the theoretical results.

2. An interface problem for Stokes equations

2.1. **The Stokes problem.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain in $d = 2$ or 3 space dimensions. As a model for the flow of an incompressible viscous fluid confined in Ω, we consider the stationary Stokes problem with homogenous Dirichlet boundary conditions

$$\left\{ \begin{array}{rcll} -\Delta \boldsymbol{u} + \boldsymbol{\nabla} p &=& \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &=& 0 & \text{in } \Omega, \\ \boldsymbol{u} &=& \boldsymbol{0} & \text{on } \partial\Omega. \end{array} \right. \tag{1}$$

In order to guarantee uniqueness of the pressure p, we assume that the pressure has mean value zero.

As function spaces for an appropriate formulation of the problem, we use

$$H_0^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\} \quad \text{and} \quad L_0^2(\Omega) := \left\{q \in L^2(\Omega) : \int_\Omega q\,dx = 0;\right\},$$

and their vector valued analogues are denoted with bold letters. For any $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, the Stokes problem (1) has a unique (weak) solution $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$; cf. [15]. The proof of this result relies on the surjectivity of the divergence operator i.e., there exists a positive constant β_Ω depending only on the domain Ω such that

$$\sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \geq \beta_\Omega \|q\|_0 \qquad \text{for all } q \in L_0^2(\Omega); \tag{2}$$

here $b(\boldsymbol{v}, q) := -\int_\Omega \operatorname{div} \boldsymbol{v}\, q\,dx$ denotes the bilinear form associated with the weak formulation of the divergence constraint.

Remark 2.1. On domains with $C^{1,1}$ boundary, the solution of (1) satisfies $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, $p \in H^1(\Omega)$, whenever $f \in \boldsymbol{L}^2(\Omega)$. The same regularity holds for convex polygonal domains in $\mathbb{R}^2$. For the derivation of various regularity results, also in L^p spaces, see [17].

2.2. **The interface problem.** Assume that Ω is partitioned into disjoint Lipschitz subdomains Ω_i such that $\bigcup_i \overline{\Omega_i} = \overline{\Omega}$. Such a partition will be denoted by $\Omega_h := \{\Omega_i : i = 1, \ldots, N\}$. The set of interior domain boundaries is defined accordingly by $\partial\Omega_h := \{\partial\Omega_i \setminus \partial\Omega : i = 1, \ldots, N\}$. By $\boldsymbol{n}_i$ we denote the unit normal vector pointing to the exterior of Ω_i. The symbol $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $i < j$ denotes the interfaces between adjacent subdomains, and the set of all interfaces is denoted $\Gamma_h := \{\Gamma_{ij} : 1 \leq i < j \leq N\}$. The union of all interfaces $\Gamma := \bigcup_{i<j} \Gamma_{ij}$ is called the *skeleton* or simply the *interface.* On Γ_{ij} we define a unique normal vector by $\boldsymbol{n}_{ij} := \boldsymbol{n}_i = -\boldsymbol{n}_j$. A sketch of such a decomposition is depicted in Figure 1.

Under the assumption that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the Stokes problem (1) can be shown to be equivalent to the following interface problem, cf. [24]:

$$\begin{cases} -\Delta\, \boldsymbol{u}_i + \boldsymbol{\nabla} p_i &= \boldsymbol{f}_i & \text{in } \Omega_i, \\ \operatorname{div} \boldsymbol{u}_i &= 0 & \text{in } \Omega_i, \\ \boldsymbol{u}_i &= \boldsymbol{0} & \text{on } \partial\Omega \cap \partial\Omega_i, \\ \boldsymbol{u}_i &= \boldsymbol{u}_j & \text{on } \Gamma_{ij}, \\ \boldsymbol{T}_i \boldsymbol{n}_{ij} &= \boldsymbol{T}_j \boldsymbol{n}_{ij} & \text{on } \Gamma_{ij}, \end{cases} \tag{3}$$

for all $\Omega_i \in \Omega_h$ and $\Gamma_{ij} \in \Gamma_h$. The symbol $\boldsymbol{T} := \boldsymbol{\nabla}\boldsymbol{u} - p\boldsymbol{I}$ denotes the stress tensor. The interface conditions ensure conservation of mass and momentum across the interface. To obtain uniqueness of the pressure, we tacitly assume again that the compatibility condition $\sum_i \int_{\Omega_i} p_i\,dx = 0$ holds.

2.3. **A variational characterization of solutions to the interface problem.** Let us introduce some further notation: The restriction of a function $v \in L^2(\Omega)$ to a subdomain Ω_i is denoted by $v_i := v|_{\Omega_i}$. For $s \geq 0$ we denote by $H^s(\Omega_h)$ the broken Sobolev space

$$H^s(\Omega_h) := \{v \in L^2(\Omega) : v_i \in H^s(\Omega_i) \text{ for all } \Omega_i \in \Omega_h\},$$

and the corresponding space of functions supported on the domain boundaries is denoted by $H^s(\partial\Omega_h)$. Vector and tensor valued function spaces are defined accordingly, and again denoted with bold symbols. For the scalar product of piecewise defined functions, we use the notation

$$(u,v)_{\Omega_h} := \sum_i (u_i,v_i)_{\Omega_i} \qquad \text{with} \quad (u_i,v_i)_{\Omega_i} := \int_{\Omega_i} u_i v_i \, dx.$$

The scalar products of functions supported on the domain boundaries are denoted by $(\cdot,\cdot)_{\partial\Omega_h}$. Also note that any function defined on Γ can be interpreted as a function defined on $\partial\Omega_h$.

We can now give the following variational characterization of solutions to the Stokes problem (1) respectively the Stokes interface problem (3).

Theorem 2.2. *Let $(\boldsymbol{u},p) \in \boldsymbol{H}^1_0(\Omega) \times L^2_0(\Omega)$ denote the solution of the Stokes problem* (1) *respectively the Stokes interface problem* (3)*, and define $\hat{\boldsymbol{u}} := \boldsymbol{u}|_\Gamma$. Then $(\boldsymbol{u},\hat{\boldsymbol{u}},p)$ is the unique solution in $\boldsymbol{H}^1_0(\Omega_h) \times \boldsymbol{H}^{1/2}_{00}(\Gamma_h) \times L^2_0(\Omega)$ of the variational problem*

$$\begin{aligned} a_h(\boldsymbol{u},\hat{\boldsymbol{u}};\boldsymbol{v},\hat{\boldsymbol{v}}) + b_h(\boldsymbol{v},\hat{\boldsymbol{v}};p) &= (\boldsymbol{f},\boldsymbol{v})_\Omega, \\ b_h(\boldsymbol{u},\hat{\boldsymbol{u}};q) &= 0, \end{aligned} \tag{4}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega_h) \cap \boldsymbol{H}^1_0(\Omega_h)$, $\hat{\boldsymbol{v}} \in \boldsymbol{H}^{1/2}_{00}(\Gamma)$, and $q \in H^1(\Omega_h) \cap L^2_0(\Omega)$. Here, $\boldsymbol{H}^{1/2}_{00}(\Gamma)$ denotes the space of traces of functions in $\boldsymbol{H}^1_0(\Omega)$ on the skeleton Γ, and the bilinear forms a_h and b_h are defined by

$$\begin{aligned} a_h(\boldsymbol{u},\hat{\boldsymbol{u}};\boldsymbol{v},\hat{\boldsymbol{v}}) &:= (\boldsymbol{\nabla u},\boldsymbol{\nabla v})_{\Omega_h} - (\partial_n \boldsymbol{u}, \boldsymbol{v}-\hat{\boldsymbol{v}})_{\partial\Omega_h} \\ &\quad - (\boldsymbol{u}-\hat{\boldsymbol{u}}, \partial_n \boldsymbol{v})_{\partial\Omega_h} + \tfrac{\alpha}{h}(\boldsymbol{u}-\hat{\boldsymbol{u}}, \boldsymbol{v}-\hat{\boldsymbol{v}})_{\partial\Omega_h}, \\ b_h(\boldsymbol{v},\hat{\boldsymbol{v}};q) &:= -(\operatorname{div}\boldsymbol{v}, q)_{\Omega_h} + (q\boldsymbol{n}, \boldsymbol{v}-\hat{\boldsymbol{v}})_{\partial\Omega_h}. \end{aligned}$$

The parameter α/h can be chosen arbitrarily.

Proof. Assume that $(\boldsymbol{u},p)$ is the solution of the Stokes problem (1). Then the stress tensor $\boldsymbol{T} = \boldsymbol{\nabla u} - p\boldsymbol{I} \in \boldsymbol{H}(\operatorname{div};\Omega)$ and the normal trace $\boldsymbol{T}_i\boldsymbol{n}_i$ is well-defined on the subdomain boundary $\partial\Omega_i$. By the Gauß-Green formula, we obtain

$$(\boldsymbol{f},\boldsymbol{v})_\Omega = (-\Delta\,\boldsymbol{u} + \boldsymbol{\nabla} p, \boldsymbol{v})_{\Omega_h} = (\boldsymbol{\nabla u} - p\boldsymbol{I}, \boldsymbol{\nabla v})_{\Omega_h} - (\partial_n \boldsymbol{u} - p\boldsymbol{n}, \boldsymbol{v})_{\partial\Omega_h}.$$

Also note that $(p\boldsymbol{I},\boldsymbol{\nabla v})_{\Omega_h} = (p, \operatorname{div}\boldsymbol{v})_{\Omega_h}$. Since $(\boldsymbol{u},p)$ solves also the interface problem (3), we further obtain that $\boldsymbol{u} - \hat{\boldsymbol{u}} \equiv 0$ on $\partial\Omega_h$, and therefore the terms involving $\boldsymbol{u}-\hat{\boldsymbol{u}}$ vanish. This already yields $a_h(\boldsymbol{u},\hat{\boldsymbol{u}};\boldsymbol{v},\boldsymbol{0}) + b_h(\boldsymbol{v},\boldsymbol{0};p) = (\boldsymbol{f},\boldsymbol{v})_\Omega$. Due to the jump condition on the normal stresses, it follows that $(\boldsymbol{Tn},\hat{\boldsymbol{v}})_{\partial\Omega_h} = (\partial_n\boldsymbol{u} - p\boldsymbol{n}, \hat{\boldsymbol{v}})_{\partial\Omega_h} = 0$, so the terms involving $\hat{\boldsymbol{v}}$ vanish as well. Finally, by the definition of $\hat{\boldsymbol{u}}$ and the continuity of $\boldsymbol{u}$ across the interfaces, we have $b_h(\boldsymbol{u},\hat{\boldsymbol{u}};q) = -(\operatorname{div}\boldsymbol{u},q) = 0$ for all test functions q.

Now assume that $(\boldsymbol{u},\hat{\boldsymbol{u}},p)$ is solution of the variational problem. Then for $\boldsymbol{v} \in \boldsymbol{C}^\infty_0(\Omega_i)$ (extended by zero to Ω), and $\hat{\boldsymbol{v}} \equiv \boldsymbol{0}$, we have

$$(\boldsymbol{f},\boldsymbol{v})_{\Omega_i} = (\boldsymbol{\nabla u},\boldsymbol{\nabla v})_{\Omega_i} - (\operatorname{div}\boldsymbol{v},p)_{\Omega_i} = (-\Delta\,\boldsymbol{u} + \boldsymbol{\nabla} p, \boldsymbol{v})_{\Omega_i},$$

from which it follows that $-\Delta\,\boldsymbol{u}_i + \boldsymbol{\nabla} p_i = \boldsymbol{f}_i$ in Ω_i. Next, for any $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega_h) \cap \boldsymbol{H}^1_0(\Omega)$ and $\hat{\boldsymbol{v}} := \boldsymbol{v}|_\Gamma$ we obtain

$$0 = a_h(\boldsymbol{u},\hat{\boldsymbol{u}};\boldsymbol{v},\hat{\boldsymbol{v}}) + b_h(\boldsymbol{v},\hat{\boldsymbol{v}};p) - (\boldsymbol{f},\boldsymbol{v})_\Omega = -(\boldsymbol{u}-\hat{\boldsymbol{u}}, \partial_n\boldsymbol{v})_{\Omega_h},$$

which implies that $\boldsymbol{u}$ is continuous across the interface, and $\hat{\boldsymbol{u}} = \boldsymbol{u}|_\Gamma$. For $\boldsymbol{v} \equiv 0$ and arbitrary choice of the test function $\hat{\boldsymbol{v}}$, we obtain

$$0 = a_h(\boldsymbol{u},\hat{\boldsymbol{u}};\boldsymbol{0},\hat{\boldsymbol{v}}) + b_h(\boldsymbol{0},\hat{\boldsymbol{v}};p) = (\partial_n\boldsymbol{u} - p\boldsymbol{n}, \hat{\boldsymbol{v}})_{\partial\Omega_h},$$

from which the continuity of the normal components of the stress tensor can be concluded. Testing with $q \in H^1(\Omega) \times L_0^2(\Omega)$, we finally obtain the divergence free condition. Therefore, any solution $(\boldsymbol{u}, \hat{\boldsymbol{u}}, p)$ of the variational problem solves the interface problem (3), and by equivalence also the Stokes problem (1). Uniqueness then follows from the uniqueness of solutions for the Stokes problem. □

Remark 2.3. The smoothness conditions on the test functions can be relaxed: e.g., it suffices to require the functions $\boldsymbol{v}$ and q to be sufficiently smooth, such that the normal traces of $\boldsymbol{\nabla v}$ and q are well-defined on the subdomain boundaries $\partial\Omega_h$. Alternatively, the test functions could be chosen from the space of solutions $(\boldsymbol{v_g}, q_{\boldsymbol{g}})$ of the Stokes problem $-\Delta\, \boldsymbol{v} + \boldsymbol{\nabla} q = \boldsymbol{g}$, $\operatorname{div} \boldsymbol{v} = 0$ with arbitrary $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$. Other generalizations will be discussed in Remark 3.13.

Remark 2.4. If the test functions $\boldsymbol{v}$ and q are locally supported on Ω_i, and $\hat{\boldsymbol{v}} \equiv 0$, then the bilinear forms a_h, b_h reduce to

$$a_i(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{\nabla u}, \boldsymbol{\nabla v})_{\Omega_i} \qquad \text{and} \qquad b_i(\boldsymbol{v}, q) = -(\operatorname{div} \boldsymbol{v}, q)_{\Omega_i},$$

which are the standard bilinear forms for the Stokes problem on the subdomains. This was already used in the second part of the previous proof.

The variational characterization of solutions stated in Theorem 2.2 will be the basis for the derivation of discrete methods for the interface problem.

3. Discretization by finite elements

In this section, we analyze the discretization of the variational problem (4) by finite element methods. For ease of presentation, we consider in the following a two dimensional model problem, and we restrict ourselves to a particular choice of finite element spaces. Generalization to three dimensions and other types of finite elements will be discussed in Section 4.

3.1. Assumptions on the domain partition and the mesh.

Throughout this section, we shall assume that Ω is a two dimensional domain, and that the subdomains Ω_i are polygonal. By $\mathcal{T}_h(\Omega_i)$ we denote quasi-uniform triangular meshes of the subdomains Ω_i, and we define the global mesh by $\mathcal{T}_h(\Omega_h) := \bigcup_i \mathcal{T}_h(\Omega_i)$. The interfaces $\Gamma_{ij} \subset \Gamma_h$ are supposed to be decomposed into segments, and the collection of the interface meshes is denoted by $\mathcal{E}_h(\Gamma_h)$. The triangulations of the subdomains and the interfaces are assumed to be of comparable size h, but otherwise they can be chosen independently from each other. An example of such a domain decomposition together with appropriate triangulations is depicted in Figure 1.

Remark 3.1. We implicitly assumed that the partition of the skeleton Γ is "conforming" in the sense that the cross-points of the domain decomposition are vertices of the interface meshes; see [25, 16] for similar conditions employed for discontinuous Galerkin discretizations. We will write $\mathcal{E}_h(\Gamma)$ instead of $\mathcal{E}_h(\Gamma_h)$ for the set of interface elements, if the domain interfaces $\Gamma_{ij} \subset \Gamma_h$ are not triangulated in a conforming manner by the interface mesh.

3.2. Discretization of the interface problem.

For the discretization of the variational problem (4), we consider finite element spaces made up of inf-sup stable elements on the

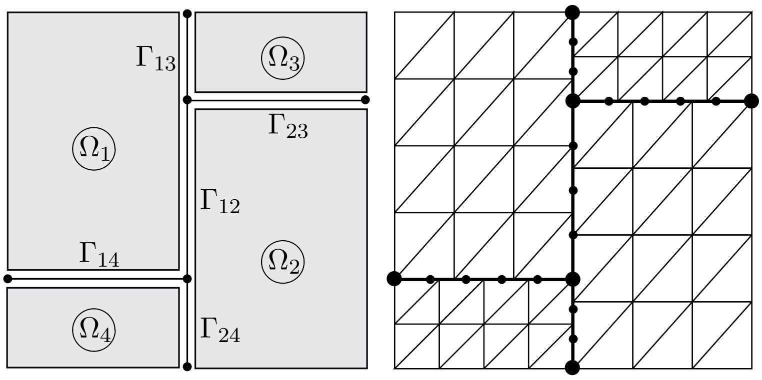

FIGURE 1. Domain partition $\Omega_h = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$ and corresponding partition of the skeleton Γ given by $\Gamma_h = \{\Gamma_{12}, \Gamma_{13}, \Gamma_{14}, \Gamma_{23}, \Gamma_{24}\}$ (left). A possible triangulation used for the first test problem in Section 5 is depicted on the right. The crosspoints of the domain partition are depicted with (bold) circles. The interface mesh $\mathcal{E}_h(\Gamma)$ is conforming in the sense of Section 3.1, i.e., the partition Γ_h of the skeleton is resolved by the interface mesh.

subdomains. In particular, we consider in this section the choice

$$\boldsymbol{V}_h := \left\{\boldsymbol{v}_h \in \boldsymbol{H}_0^1(\Omega_h) : \boldsymbol{v}_h|_T \in [\mathcal{P}^2(T)]^2 \quad \text{for all} \quad T \in \mathcal{T}_h(\Omega_h)\right\},$$
$$Q_h := \left\{q_h \in L_0^2(\Omega) : q_h|_T \in \mathcal{P}^0(T) \quad \text{for all} \quad T \in \mathcal{T}_h(\Omega_h)\right\},$$

and we utilize discontinuous piecewise quadratic functions on the skeleton, i.e.,

$$\widehat{\boldsymbol{V}}_h := \left\{\hat{\boldsymbol{v}}_h \in \boldsymbol{L}^2(\Gamma_h) : \hat{\boldsymbol{v}}_h|_E \in [\mathcal{P}^2(E)]^2 \quad \text{for all} \quad E \in \mathcal{E}_h(\Gamma_h)\right\}.$$

As usual, $\mathcal{P}^k(T)$ denotes the space of polynomials of maximal order k on the element T.

Remark 3.2. Note that the space $\widehat{\boldsymbol{V}}_h$ is not a subspace of $H_{00}^{1/2}(\Gamma)$ used in the variational characterization of Theorem 2.2. Therefore, this discretization is non-conforming, and an appropriate analysis based on mesh dependent norms has to be used.

The spaces $\boldsymbol{V}_h$, Q_h can be considered to be the natural generalization of the $P_2 - P_0$ element to the interface problem under consideration. The spaces

$$\boldsymbol{V}_h(\Omega_i) := \boldsymbol{V}_h|_{\Omega_i} \cap \boldsymbol{H}_0^1(\Omega_i) := \left\{\boldsymbol{v}_h \in \boldsymbol{H}_0^1(\Omega_i) : \boldsymbol{v}_h|_T \in [\mathcal{P}^2(T)]^2 \quad \forall\, T \in \mathcal{T}_h(\Omega_i)\right\},$$
$$Q_h(\Omega_i) := Q_h|_{\Omega_i} \cap L_0^2(\Omega_i) := \left\{q_h \in L_0^2(\Omega_i) : q_h|_T \in \mathcal{P}^0(T) \quad \forall\, T \in \mathcal{T}_h(\Omega_i)\right\},$$

form an inf-sup stable pair for the Stokes problem on the subdomain Ω_i, i.e., a discrete analogue of (2) holds. For later reference, let us recall the following result [15].

Lemma 3.3. *The spaces $\boldsymbol{V}_h(\Omega_i)$, $Q_h(\Omega_i)$ satisfy the discrete inf-sup condition*

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega_i)} \frac{b_i(\boldsymbol{v}_h, p_h)}{\|\boldsymbol{v}_h\|_{1,\Omega_i}} \geq \beta_i \|p_h\|_{0,\Omega_i} \qquad \forall\, p_h \in Q_h(\Omega_i),$$

for some $\beta_i > 0$ independent of h.

For discretization of the interface problem, we now consider the following hybrid mortar finite element method.

Method 3.1. *Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, find $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h, p_h) \in \boldsymbol{V}_h \times \widehat{\boldsymbol{V}}_h \times Q_h$ such that*

$$a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) + b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p_h) = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega,$$
$$b_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; q_h) = 0,$$

holds for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{v}}_h \in \widehat{\boldsymbol{V}}_h$, and $q_h \in Q_h$. The bilinear forms a_h and b_h are defined as in Theorem 2.2. The parameter h now denotes the local meshsize, and $\alpha > 0$ will be chosen below.

Remark 3.4. Method 3.1 generalizes the hybrid mortar methods for elliptic equations proposed in [10]. We employ the results obtained for hybrid mortaring of elliptic problems [11] to prove ellipticity of the bilinear form a_h below. The proof of the inf-sup stability of the bilinear form b_h requires different arguments.

3.3. **Analysis on the discrete level.** For the stability analysis of the hybrid mortar finite element method 3.1, we will utilize the following mesh dependent energy norms

$$\|(\boldsymbol{v}, \hat{\boldsymbol{v}})\|_{1,h} := \left(\|\boldsymbol{\nabla} \boldsymbol{v}\|^2_{\Omega_h} + \alpha|\boldsymbol{v} - \hat{\boldsymbol{v}}|^2_{1/2,h}\right)^{1/2} \qquad \text{and} \qquad \|q\|_{0,h} := \|q\|_{\Omega_h},$$

where $\|u\|_{\Omega_h} := (u, u)^{1/2}_{\Omega_h}$ and $|u|_{\partial\Omega_h} := (u, u)^{1/2}_{\partial\Omega_h}$. The expressions

$$|\boldsymbol{v}|^2_{1/2,h} := h^{-1}|\boldsymbol{v}|^2_{\partial\Omega_h} \quad \text{and} \quad |\boldsymbol{v}|^2_{-1/2,h} := h|\boldsymbol{v}|^2_{\partial\Omega_h}.$$

denote discrete trace norms. The stabilization parameter $\alpha > 0$ will be chosen later, but it is the same as in the definition of a_h. Similar norms are frequently used in the analysis of discontinuous Galerkin finite element methods, cf. e.g., [1] and [7] for the norms of functions supported on the subdomain boundaries.

Below, we will use the following discrete trace inequality, which follows readily from the standard scaling arguments.

Lemma 3.5. *There exists a mesh independent constant C_t, such that*

$$|\partial_n \boldsymbol{v}_h|_{-1/2,\Omega_i} \leq C_t \, \|\boldsymbol{\nabla} \boldsymbol{v}_h\|_{0,\Omega_h},$$

for all functions $\boldsymbol{v}_h \in \boldsymbol{V}_h$.

Estimates of the constant C_t are derived for the case of piecewise linear functions in [3], but also the dependence on the polynomial degree can be made explicit [23, 18].

In the following, we establish the conditions needed for the application of Brezzi's theorem, which will guarantee existence and uniqueness of a finite element solution for the discrete variational problem. Let us start with showing coercivity and boundedness of the bilinear form a_h on the finite element spaces.

Proposition 3.6. *Let $\alpha \geq 2C_t^2$. Then, for all $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h) \in \boldsymbol{V}_h \times \widehat{\boldsymbol{V}}_h$, there holds*

$$a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{u}_h, \hat{\boldsymbol{u}}_h) \geq \tfrac{1}{2}\|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|^2_{1,h}.$$

Proof. To keep track of the constants, and for later reference, let us recall the short proof from [11]. Using the definition of a_h, and Young's inequality, we obtain

$$\begin{aligned} a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{u}_h, \hat{\boldsymbol{u}}_h) &= \|\boldsymbol{\nabla} \boldsymbol{u}_h\|^2_{\Omega_h} - 2\,(\partial_n \boldsymbol{u}_h, \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h)_{\partial\Omega_h} + \alpha|\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h|^2_{1/2,h} \\ &\geq \|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|^2_{1,h} - \tfrac{2}{\alpha}|\partial_n \boldsymbol{u}_h|^2_{-1/2,h} - \tfrac{\alpha}{2}|\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h|^2_{1/2,h} \\ &\geq \min(1 - C_t^2/\alpha, \tfrac{1}{2})\|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|^2_{1,h} = \tfrac{1}{2}\|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|^2_{1,h}. \end{aligned}$$

For the last steps, we used the discrete trace inequality and the condition on α. $\square$

Proposition 3.7. *For all $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{v}}_h \in \widehat{\boldsymbol{V}}_h$, and $p_h, q_h \in Q_h$ there holds*

$$a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{u}_h, \hat{\boldsymbol{v}}_h) \leq C_a \|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} \|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h},$$
$$b_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; p_h) \leq C_b \|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} \|p_h\|_{0,h},$$

with constants C_a, C_b independent of the meshsize h.

Proof. The result follows from the Cauchy-Schwarz inequality, the definition of the norms, and the discrete trace inequality. $\square$

It remains to establish a discrete inf-sup stability condition for the bilinear form b_h. For the proof of the corresponding result, we will also require the following Lemma.

Lemma 3.8. *Let $\widehat{\Pi}_h : \boldsymbol{H}^1(\Omega) \to \widehat{\boldsymbol{V}}_h$ be the orthogonal projector with respect to $L^2(\Gamma)$. Then*

$$|\widehat{\Pi}_h \boldsymbol{v}|_{1/2,h} \leq C_{\widehat{\Pi}} \|\boldsymbol{v}\|_{1,\Omega} \qquad \text{and} \qquad (\widehat{\Pi}_h \boldsymbol{v}, \boldsymbol{n})_{\Gamma_{ij}} = (\boldsymbol{v}, \boldsymbol{n})_{\Gamma_{ij}}$$

for some constant $C_{\widehat{\Pi}}$, all $\Gamma_{ij} \in \Gamma_h$ and all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$.

Proof. The result follows from the trace inequality and the standard scaling arguments. $\square$

The following results plays the central role in this section.

Theorem 3.9. *The bilinear form b_h satisfies a discrete inf-sup condition, i.e., there exists a constant $\beta > 0$ independent of h such that*

$$\sup_{(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)} \frac{b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p_h)}{\|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h}} \geq \beta \|p_h\|_0,$$

holds uniformly for all functions $p_h \in Q_h$.

Proof. For the proof of this result, we use an argument introduced by Boland and Nicolaides [6] to explicitly construct a pair of functions $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)$ that satisfies the inequality.

Step 1: Any function $p_h \in Q_h$ can be decomposed into

$$p_h = p_{a,h} + p_{b,h},$$

where the function $p_{b,h}$ is constant on each subdomain, and $p_{a,h}$ has zero mean on each subdomain, i.e., $p_{a,h}|_{\Omega_i} \in L^2_0(\Omega_i)$. Note that the two functions are orthogonal with respect to the scalar product of $L^2(\Omega_h)$.

Step 2: By Lemma 3.3, we can define a function $\boldsymbol{u}_{a,h} \in \boldsymbol{V}_h$ independently on each subdomain, such that $\boldsymbol{u}_{a,h}|_{\Omega_i} \in \boldsymbol{H}^1_0(\Omega_i)$, and moreover

$$-(\operatorname{div} \boldsymbol{u}_{a,h}, p_{a,h})_{\Omega_i} = \|p_{a,h}\|^2_{\Omega_i} \qquad \text{and} \qquad \|\boldsymbol{u}_{a,h}\|_{1,\Omega_i} \leq \beta_i^{-1} \|p_{a,h}\|_{\Omega_i}.$$

Setting $\hat{\boldsymbol{u}}_{a,h} := \boldsymbol{0}$, this yields

$$b_h(\boldsymbol{u}_{a,h}, \hat{\boldsymbol{u}}_{a,h}; p_{a,h}) = \|p_{a,h}\|^2_{0,h} \qquad \text{and} \qquad \|(\boldsymbol{u}_{a,h}, \hat{\boldsymbol{u}}_{a,h})\|_{1,h} \leq \beta_a^{-1} \|p_{a,h}\|_{0,h}.$$

with constant β_a defined by $\beta_a^{-1} := \max_i \beta_i^{-1}$.

Step 3: By the surjectivity of the divergence operator (2), there exists a function $\boldsymbol{u}_b \in \boldsymbol{H}^1_0(\Omega)$ such that

$$-\operatorname{div} \boldsymbol{u}_b = p_{b,h} \qquad \text{and} \qquad \|\boldsymbol{u}_b\|_{1,\Omega} \leq \beta_\Omega^{-1} \|p_{b,h}\|_{0,h}.$$

Defining $\boldsymbol{u}_{b,h} := \mathbf{0}$ and $\hat{\boldsymbol{u}}_{b,h} := \widehat{\Pi}_h \boldsymbol{u}_b$, with $\widehat{\Pi}_h$ from Lemma 3.8, now yields

$$b_h(\boldsymbol{u}_{b,h}, \hat{\boldsymbol{u}}_{b,h}; p_{b,h}) = \|p_{b,h}\|_{0,h}^2 \qquad \text{and} \qquad \|(\boldsymbol{u}_{b,h}, \hat{\boldsymbol{u}}_{b,h})\|_{1,h} \leq \beta_b^{-1} \|p_{b,h}\|_{0,h},$$

with constant β_b defined by $\beta_b^{-1} := \beta_\Omega^{-1} C_{\widehat{\Pi}} / \sqrt{\alpha}$.

Step 4: Let us define $\boldsymbol{u}_h := \boldsymbol{u}_{a,h} + \varepsilon \boldsymbol{u}_{b,h}$ for $\epsilon > 0$, and set $\hat{\boldsymbol{u}}_h := \hat{\boldsymbol{u}}_{a,h} + \varepsilon \hat{\boldsymbol{u}}_{b,h}$. Since by construction, $p_{b,h}$ is constant on each subdomain, we obtain $b_h(\boldsymbol{u}_{a,h}, \hat{\boldsymbol{u}}_{a,h}; p_{b,h}) = 0$, and consequently

$$\begin{aligned} b_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; p_h) &= b_h(\boldsymbol{u}_{a,h}, \hat{\boldsymbol{u}}_{a,h}; p_{a,h}) + \varepsilon\, b_h(\boldsymbol{u}_{b,h}, \hat{\boldsymbol{u}}_{b,h}; p_{b,h}) + \varepsilon\, b_h(\boldsymbol{u}_{b,h}, \hat{\boldsymbol{u}}_{b,h}; p_{a,h}) \\ &\geq \|p_{a,h}\|_{0,h}^2 + \varepsilon \|p_{b,h}\|_{0,h}^2 - \varepsilon C_b \|(\boldsymbol{u}_{b,h}, \hat{\boldsymbol{u}}_{b,h})\|_{1,h} \|p_{a,h}\|_{0,h} \\ &\geq \tfrac{1}{2} \|p_{a,h}\|_{0,h}^2 + \varepsilon \left(1 - \varepsilon C_b^2 \beta_b^{-2}/2\right) \|p_{b,h}\|_{\Omega_h}^2. \end{aligned}$$

Choosing $\varepsilon = \beta_b^2 / C_b^2$, and utilizing the orthogonality of $p_{a,h}$ and $p_{b,h}$, this yields $b_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; p_h) \geq \frac{1}{2} \min(1, \beta_b^2 C_b^{-2}) \|p_h\|_{0,h}^2$. From the norm estimates for $\boldsymbol{u}_{a,h}$ and $\boldsymbol{u}_{b,h}$, we finally obtain

$$\|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} \leq \max(\beta_a^{-1}, \beta_b C_b^{-2}) \|p_h\|_{0,h},$$

and the assertion of the theorem follows, e.g., with $\beta = \frac{1}{2} \min(1, \beta_b^2 C_b^{-2}) \min(\beta_a, \beta_b^{-1} C_b^2)$. □

Brezzi's theorem [8] now implies existence and uniqueness of a finite element solution.

Theorem 3.10. *Method 3.1 has a unique solution* $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h, p_h) \in \boldsymbol{V}_h \times \widehat{\boldsymbol{V}}_h \times Q_h$ *that satisfies the a-priori estimate*

$$\|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} + \|p\|_{0,h} \leq C \|f\|_{\Omega_h},$$

with a constant C *independent of the mesh and the data.*

A direct consequence of Theorem 3.10 is the discrete stability in the sense of Babuška-Aziz.

Corollary 3.11. *For any* $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h, p_h) \in \boldsymbol{V}_h \times \widehat{\boldsymbol{V}}_h \times Q_h$ *there exist a triple of functions* $(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h, q_h) \in \boldsymbol{V}_h \times \widehat{\boldsymbol{V}}_h \times Q_h$ *such that*

$$\begin{aligned} a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) &+ b_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; q_h) + b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p_h) \\ &\geq c_{stab} (\|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} + \|p_h\|_{0,h}) (\|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h} + \|q_h\|_{0,h}), \end{aligned}$$

with a constant c_{stab} *independent of the meshsize* h *and the functions* $\boldsymbol{u}_h$, $\hat{\boldsymbol{u}}_h$*), and* p_h.

3.4. **A-priori error estimates.** For obtaining a-priori error bounds, we require additional properties of the bilinear forms, and some interpolation error estimates.

Lemma 3.12. *Let* $(\boldsymbol{u}, p)$ *denote the solution of the interface problem* (3)*, and additionally assume that* $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega_h)$ *and* $p \in H^1(\Omega_h)$*. Then Galerkin orthogonality holds, i.e., if* $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h, p_h)$ *denotes the solution of Method 3.1, then*

$$\begin{aligned} a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) + b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p - p_h) &= 0, \\ b_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \hat{\boldsymbol{u}}_h; q_h) &= 0, \end{aligned}$$

for all (discrete) test functions $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{v}}_h \in \widehat{\boldsymbol{V}}_h$*, and* $q_h \in Q_h$.

Remark 3.13. For conditions on the domain Ω providing sufficient regularity of the solution $(\boldsymbol{u}, p)$, see Remark 2.1. The regularity requirements on the solution $(\boldsymbol{u}, p)$ of the continuous problem can however be relaxed: It is possible to redefine the bilinear forms a_h and b_h, such that $\tilde{a}_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) = a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)$ and $\tilde{b}_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h, q_h) = b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; q_h)$; but additionally, $\tilde{a}_h$ and $\tilde{b}_h$ are well-defined and continuous for arguments $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, $\hat{\boldsymbol{u}} = \boldsymbol{u}|_{\Gamma_h}$, and $p \in L^2_0(\Omega)$ when tested with finite element functions $\boldsymbol{v}_h$, $\hat{\boldsymbol{v}}_h$, and q_h. In this way, the extra regularity assumptions can be dropped (also in the following results). For details about such extensions based on local liftings, we refer to [23, 20].

As a next step, we will show boundedness of the bilinear forms with respect to the slightly stronger norms

$$\begin{aligned} |||(\boldsymbol{u}, \hat{\boldsymbol{u}})|||_{1,h} &:= \big(\|(\boldsymbol{u}, \hat{\boldsymbol{u}})\|_{1,h}^2 + |\partial_n \boldsymbol{u}|_{-1/2,h}^2\big)^{1/2}, \\ |||p|||_{0,h} &:= \big(\|p\|_{0,h}^2 + |p|_{-1/2,h}^2\big)^{1/2}. \end{aligned}$$

Note that on the finite dimensional spaces $\boldsymbol{V}_h$, $\widehat{\boldsymbol{V}}_h$, and Q_h, the norms $\|(\cdot,\cdot)\|_{1,h}$ and $|||(\cdot,\cdot)|||_{1,h}$, respectively $\|\cdot\|_{0,h}$ and $|||\cdot|||_{0,h}$ are equivalent.

Lemma 3.14. *There exist positive constants $\widetilde{C}_a$ and $\widetilde{C}_b$ independent of the mesh size h, such that the bounds*

$$a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \hat{\boldsymbol{u}}_h; \boldsymbol{u}_h, \hat{\boldsymbol{v}}_h) \le \widetilde{C}_a |||(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \hat{\boldsymbol{u}}_h)|||_{1,h} \|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h},$$

and

$$\begin{aligned} b_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \hat{\boldsymbol{u}}_h; q_h) &\le \widetilde{C}_b |||(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \hat{\boldsymbol{u}}_h|||_{1,h} \|q_h\|_{0,h}, \\ b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p - p_h) &\le \widetilde{C}_b \|(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \hat{\boldsymbol{u}}_h\|_{1,h} |||p - p_h|||_{0,h}, \end{aligned}$$

hold for for all $\boldsymbol{u}_h$, $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{u}}_h$, $\hat{\boldsymbol{v}}_h \in \widehat{\boldsymbol{V}}_h$, p_h, $q_h \in Q_h$, and all $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega_h)$, $p \in \boldsymbol{H}^1(\Omega_h)$.

Proof. The result follows directly by application of the Cauchy-Schwarz inequality, the definition of the norms, and the equivalence of the energy norms $\|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h}$, $\|q_h\|_{0,h}$ and $|||(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)|||_{1,h}$, $|||q_h|||_{0,h}$ for finite element functions. □

As a final ingredient for the error analysis, let us characterize the approximation properties of the finite element spaces with respect to the norms $|||(\cdot,\cdot)|||_{1,h}$ and $|||\cdot|||_{0,h}$: By $\Pi_h : \boldsymbol{H}^1(\Omega_h) \to \boldsymbol{V}_h$ we denote an interpolation operator which is defined subdomain wise by a standard interpolation procedure, e.g., a Clément or Scott-Zhang operator. Moreover, let $\widehat{\Pi}_h : \boldsymbol{H}^1(\Gamma_h) \to \widehat{\boldsymbol{V}}_h$ denote the projection of functions on the interface with respect to the L^2-norm, and by $\Pi_0 : L^2_0(\Omega) \to Q_h$ the projection onto piecewise constants on the elements. For these interpolation operators, the following standard interpolation error estimates hold.

Lemma 3.15. *For any function $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega_h) \times \boldsymbol{H}^1_0(\Omega)$ there holds the estimate*

$$|||(\boldsymbol{u} - \Pi_h \boldsymbol{u}, \boldsymbol{u} - \widehat{\Pi}_h \boldsymbol{u})|||_{1,h} + h^{-1}\|u - \Pi_h \boldsymbol{u}\|_{0,h} \le Ch\|u\|_{2,\Omega_h}, \tag{5}$$

with a constant C independent of h. For any $p \in H^1(\Omega_h) \cap L^2_0(\Omega_h)$, there holds

$$|||p - \Pi_0 p|||_{0,h} \le Ch\|p\|_{1,\Omega_h}, \tag{6}$$

with a (maybe different) meshsize independent constant C.

Proof. The estimates follow with the usual scaling arguments. □

Combining the discrete stability result of Corollary 3.11 with the boundedness and the interpolation error estimates, we obtain the following error bound in the energy norm.

Theorem 3.16. *Let* $(\boldsymbol{u}, p)$ *denote the solution of* (1) *and assume that* $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega_h)$ *and* $p \in H^1(\Omega_h)$. *Moreover, let* $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h, p_h)$ *be the finite element solution of Method 3.1. Then*

$$\|(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \hat{\boldsymbol{u}}_h)\|_{1,h} + \|p - p_h\|_{0,h} \leq Ch\left(\|\boldsymbol{u}\|_{2,\Omega_h} + \|p\|_{1,\Omega_h}\right)$$

with a constant C *independent of the meshsize* h.

Proof. By Corollary 3.11, Lemma 3.12, and Lemma 3.14, we obtain

$$\begin{aligned} c_{stab}(\|(\boldsymbol{u}_h - \Pi_h u, \hat{\boldsymbol{u}}_h - \widehat{\Pi}_h \boldsymbol{u})\|_{1,h} + \|p_h - \Pi_0 p\|_{0,h}) \\ \leq (\widetilde{C}_a + \widetilde{C}_b) |\!|\!|(\boldsymbol{u} - \Pi_h \boldsymbol{u}, \boldsymbol{u} - \widehat{\Pi}_h \boldsymbol{u}|\!|\!|_{1,h} + |\!|\!|p - \Pi_0 p|\!|\!|_{0,h}, \end{aligned}$$

from which the estimate follows via the triangle inequality and the interpolation error estimates. □

Provided that the continuous problem is sufficiently regular, we also obtain optimal error estimates with respect to the L^2-norm.

Theorem 3.17. *Let* $(\boldsymbol{u}, p) \in H^2(\Omega_h) \times H^1(\Omega_h)$ *be the solution of* (1), *and* $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h, p_h)$ *be the solution of Method 3.1. If for any* $\boldsymbol{f} \in L^2(\Omega)$, *the solution* $(\boldsymbol{u}, p)$ *of problem* (1) *is in* $\boldsymbol{H}^2(\Omega_h) \times H^1(\Omega_h)$, *then there exists a constant* C *independent of* h, *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} + h\,\|p - p_h\|_{0,\Omega} \leq Ch^2\left(\|\boldsymbol{u}\|_{2,\Omega_h} + \|p\|_{1,\Omega_h}\right).$$

Proof. The proof follows with the standard duality argument of Aubin and Nitsche, and is therefore omitted. □

4. Generalizations and further remarks

In this section, we would like to discuss generalizations of our results in several directions, in particular, we want to consider two and three dimensional problems and discretizations based on more general inf-sup stable finite element pairs. Additionally, we intend to further relax the conditions on the domain partition and the meshes.

To avoid notational difficulties, we still assume that the subdomains Ω_i and the skeleton Γ can be meshed exactly. As for the model problem discussed in the previous section, we consider finite element spaces $\boldsymbol{V}_h$, Q_h made up of inf-sup stable finite element pairs on the subdomains, and we denote by $\boldsymbol{V}_h(\Omega_i) := \boldsymbol{V}_h|_{\Omega_i} \cap H_0^1(\Omega_i)$ and $Q_h(\Omega_i) := Q_h|_{\Omega_i} \cap L_0^2(\Omega_i)$ the restrictions of the global spaces to the subdomains. For the analysis of this section, we require the following natural stability condition.

Assumption 4.1. *The local spaces* $\boldsymbol{V}_h(\Omega_i)$, $Q_h(\Omega_i)$ *satisfy a discrete inf-sup condition, i.e., there exist constants* β_i *independent of* h *such that*

$$\text{(A1)} \qquad \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega_i)} \frac{b_i(\boldsymbol{v}_h, p_h)}{\|\boldsymbol{v}_h\|_{1,\Omega_i}} \geq \beta_i \|p_h\|_{0,\Omega_i} \qquad \textit{for all} \quad p_h \in Q_h(\Omega_i).$$

Inf-sup stable finite element pairs satisfying (A1) are well-known for triangular and rectangular meshes in 2D, and tetrahedral, or hexahedral meshes in 3D. The combination of appropriate elements allows to consider also hybrid meshes containing different element types. For a comprehensive overview over frequently used finite elements and further references, we refer to [15, 8].

Since the ellipticity of the bilinear form a_h does not depend on the choice of the interface space $\widehat{\boldsymbol{V}}_h$ [11], this space can be chosen with great flexibility. In contrast to elliptic problems, we however have to impose the following weak condition, which is needed for verification of the global inf-sup condition for the bilinear form b_h.

Assumption 4.2. *There exists an operator $\widehat{\Pi}_h : \boldsymbol{H}_0^1(\Omega) \to \widehat{\boldsymbol{V}}_h$ and a constant $C_{\widehat{\Pi}}$ such that*

$$|\widehat{\Pi}_h \boldsymbol{v}|_{1/2,h} \leq C_{\widehat{\Pi}} \|\boldsymbol{v}\|_{1,\Omega}, \tag{A2}$$

$$(\widehat{\Pi}_h \boldsymbol{v}, \boldsymbol{n})_{\Gamma_{ij}} = (\boldsymbol{v}, \boldsymbol{n})_{\Gamma_{ij}}, \tag{A3}$$

hold for all functions $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and all interfaces $\Gamma_{ij} \in \Gamma_h$.

Remark 4.3. Condition (A2) is satisfied by any reasonable interpolation or projection operator, whereas the condition (A3) requires that $\widehat{\boldsymbol{V}}_h$ is sufficiently rich. This is not surprising, since increasing the function space $\widehat{\boldsymbol{V}}_h$ helps to increase the supremum in the inf-sup condition. As a rule of thumb, it suffices that at least one degree of freedom is available for each part Γ_{ij} of the skeleton. Examples of suitable criteria for the choice of the interface space will be given in the following. We think that our requirements on the meshes are weaker than the conditions used in other works [16, 13]. In particular, the subdomain meshes can be chosen without any restrictions.

Example 4.1. Assume that the interfaces Γ_{ij} are resolved by the interface mesh, and that $\widehat{\boldsymbol{V}}_h$ contains piecewise constants. Then the conditions (A2)–(A3) are satisfied.

Example 4.2. Assume that the size of the interfaces Γ_{ij} is large compared to the meshsize h, i.e., such that the area covered by interface elements lying in the interior of Γ_{ij} is comparable to the area of Γ_{ij}. Then it is possible to define $\hat{\boldsymbol{v}}_{ij}$ with local support in Γ_{ij} such that (A2)–(A3) hold.

Example 4.3. The validity of assumptions (A2)–(A3) can also be verified algorithmically by solving local problems on the interface Γ_h. If (A2)–(A3) is not satisfied, then the space $\widehat{\boldsymbol{V}}_h$ can be enriched with characteristic functions supported on Γ_{ij}.

Remark 4.4. The discretization of the interface may consist of more general than finite element functions. For computations it might be advantageous to use splines with higher continuity or even functions with global support [19]. Our analysis also covers this case.

The generalization of the results of Section 3 is now straightforward.

Theorem 4.5. *Let the assumptions (A1)–(A3) hold. Then there exists a constant $\beta > 0$ independent of h such that*

$$\sup_{(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)} \frac{b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p_h)}{\|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h}} \geq \beta \, \|p_h\|_{0,h},$$

holds for all $p_h \in Q_h$.

Proof. The proof is identical to that of Theorem 3.9 and is therefore omitted. □

The ellipticity estimate of Proposition 3.6, and the bounds of Proposition 3.7 hold for general choices of spaces $\boldsymbol{V}_h$, $\widehat{\boldsymbol{V}}_h$. Thus, under Assumptions (A1)–(A3), the discrete method has a unique solution, and satisfies the stability condition of Corollary 3.11.

As a final ingredient, let us characterize the approximation properties of the discrete spaces.

Assumption 4.6. *Suppose that there exist interpolation operators* $\Pi_{\boldsymbol{V}_h} : \boldsymbol{H}^{k+1}(\Omega_h) \to \boldsymbol{V}_h$, $\Pi_{\widehat{\boldsymbol{V}}_h} : \boldsymbol{H}^{k+1}(\Omega_h) \to \widehat{\boldsymbol{V}}_h$, *and* $\Pi_{Q_h} : H^k(\Omega_h) \to Q_h$ *such that the estimates*

$$\text{(A4)} \qquad \|\!|(\boldsymbol{u} - \Pi_{\boldsymbol{V}_h}\boldsymbol{u}, \boldsymbol{u} - \Pi_{\widehat{\boldsymbol{V}}_h}\boldsymbol{u}\|\!|_{1,h} + h^{-1}\|\boldsymbol{u} - \Pi_{\boldsymbol{V}_h}\|_{\Omega_h} \le Ch^k\|u\|_{H^{k+1}(\Omega_h)}$$

$$\text{(A5)} \qquad \|p - \Pi_{Q_h}p\|_{0,h} \le Ch^k\|p\|_{H^k(\Omega_h)}$$

hold for all $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(\Omega_h)$ *and all* $p \in \boldsymbol{H}^k(\Omega_h)$ *with a constant* C *independent of* h.

The following a-priori estimates in the energy and L^2-norm now follow with the same arguments as in the previous section.

Theorem 4.7. *Let* $(\boldsymbol{u}, p)$ *denote the solution of the Stokes problem* (1), *and additionally assume that* $(\boldsymbol{u}, p) \in \boldsymbol{H}^{k+1}(\Omega_h) \times H^k(\Omega_h)$. *If (A1)–(A5) hold, then the solution* $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h, p_h)$ *of Method 3.1 satisfies*

$$\|(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \hat{\boldsymbol{u}}_h)\|_{1,h} + \|p - p_h\|_{0,h} \le Ch^k \left(\|\boldsymbol{u}\|_{k+1,\Omega_h} + \|p\|_{k,\Omega_h}\right)$$

with some constant C *independent of* h.

If, moreover, the continuous problem is H^2*-regular, i,.e., for any* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *the solution* $(\boldsymbol{u}, p)$ *of* (1) *is in* $\boldsymbol{H}^2(\Omega_h) \times H^1(\Omega_h)$, *then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\Omega_h} + h\|p - p_h\|_{0,h} \le Ch^{k+1} \left(\|\boldsymbol{u}\|_{k+1,\Omega_h} + \|p\|_{k,\Omega_h}\right).$$

Proof. The proofs of Theorem 3.16 and 3.17 can be applied directly. □

At the end of this section, let us make some remarks concerning further generalizations.

Remark 4.8. The assumption of quasi-uniformity of the meshes was made for notational convenience, and with obvious modifications, our results also hold for shape-regular triangulations.

Remark 4.9. The results in this paper were derived under the assumption that the interface is resolved exactly. We think, that a generalization of our results to geometrically non-matching domain partitions is possible; cf. [21] and the references therein for results in this direction. Inexact integration and other variational crimes can supposedly be incorporated as well. Such generalizations however deserve some further detailed considerations.

Remark 4.10. The generalization of our results to other boundary conditions is straight forward. Since we are already using variational arguments for dealing with the interface conditions, it seems natural to incorporate also the boundary conditions weakly, e.g., by Nitsche's method or hybrid variants. Some results in this directions are contained in [14].

Remark 4.11. Our analysis also applies to very fine domain partitions and covers, as a limiting case, also hybrid versions of discontinuous Galerkin methods; see [22] for related methods. The conditions (A1)–(A3) can easily be verified if the global mesh is generated from a conforming triangulation (eventually with hanging nodes).

5. Numerical Results

In this section, we present some results of numerical experiments for two test cases: In a first series of computations, we make a convergence study for hybrid mortar methods using different finite element discretizations. As a second problem, we consider the backward facing step flow, which serves as a benchmark problem. For this second test case, we compare the results obtained with the hybrid mortar method on non-matching triangulations to those obtained with continuous finite elements for conforming meshes.

All numerical results were obtained with a finite element code using the Dune framework; in particular, we used the Grid-Glue module [2] for handling of the non-matching interfaces.

5.1. **Convergence studies.** The first test problem is a simple model of a *colliding flow* on a square domain $\Omega = (-1,1)^2$. Boundary conditions are chosen, such that the exact solution is given by

$$\boldsymbol{u} = \left(20xy^3, 5x^4 - 5y^4\right), \quad p = 60x^2y - 20y^3.$$

The domain is partitioned into four subdomains $\Omega_1 = (-1,0) \times (-0.5,1)$, $\Omega_2 = (0,1) \times (-1,0.5)$, $\Omega_3 = (0,1) \times (0.5,1)$ and $\Omega_4 = (-1,0) \times (-1,-0.5)$; see Figure 1 for a sketch of the domain partition and the initial mesh. The hybrid mortar method is applied on a sequence of uniformly refined meshes, and the discretization errors in the L^2-norm and the energy norm are computed using the analytic solution. The results for different inf-sup stable approximations are listed in Table 5.1. For discretization of the interface space, we utilize piecewise (discontinuous) polynomials of order k, where $k = 2,3$ is the order of the velocity approximation.

h	$\alpha = 5.0$				$\alpha = 20.0$			
$\mathbf{P_2 - P_0}$	L^2	rate	energy	rate	L^2	rate	energy	rate
1.0	$7.635 \cdot 10^{0}$	–	$1.332 \cdot 10^{1}$	–	$7.616 \cdot 10^{0}$	–	$1.318 \cdot 10^{1}$	–
0.5	$1.938 \cdot 10^{0}$	1.977	$6.912 \cdot 10^{0}$	0.946	$1.937 \cdot 10^{0}$	1.975	$6.876 \cdot 10^{0}$	0.939
0.25	$4.854 \cdot 10^{-1}$	1.997	$3.507 \cdot 10^{0}$	0.979	$4.853 \cdot 10^{-1}$	1.997	$3.497 \cdot 10^{0}$	0.975
0.125	$1.213 \cdot 10^{-1}$	1.999	$1.763 \cdot 10^{0}$	0.991	$1.213 \cdot 10^{-1}$	1.999	$1.760 \cdot 10^{0}$	0.990
$\mathbf{P_2 - P_1}$	L^2	rate	energy	rate	L^2	rate	energy	rate
1.0	$1.419 \cdot 10^{0}$	–	$3.255 \cdot 10^{0}$	–	$1.453 \cdot 10^{0}$	–	$3.183 \cdot 10^{0}$	–
0.5	$1.710 \cdot 10^{-1}$	3.052	$7.615 \cdot 10^{-1}$	2.095	$1.727 \cdot 10^{-1}$	3.072	$7.443 \cdot 10^{-1}$	2.096
0.25	$2.115 \cdot 10^{-2}$	3.015	$1.830 \cdot 10^{-1}$	2.056	$2.120 \cdot 10^{-2}$	3.026	$1.797 \cdot 10^{-1}$	2.050
0.125	$2.636 \cdot 10^{-3}$	3.003	$4.476 \cdot 10^{-2}$	2.031	$2.639 \cdot 10^{-3}$	3.006	$4.423 \cdot 10^{-2}$	2.022
$\mathbf{P_3 - P_2}$	L^2	rate	energy	rate	L^2	rate	energy	rate
1.0	$9.871 \cdot 10^{-2}$	–	$2.586 \cdot 10^{-1}$	–	$1.000 \cdot 10^{-1}$	–	$2.187 \cdot 10^{-1}$	–
0.5	$5.925 \cdot 10^{-3}$	4.058	$2.981 \cdot 10^{-2}$	3.116	$5.982 \cdot 10^{-3}$	4.063	$2.673 \cdot 10^{-2}$	3.032
0.25	$3.734 \cdot 10^{-4}$	3.988	$3.550 \cdot 10^{-3}$	3.070	$3.739 \cdot 10^{-4}$	3.999	$3.306 \cdot 10^{-3}$	3.015
0.125	$2.416 \cdot 10^{-5}$	3.950	$4.346 \cdot 10^{-4}$	3.030	$2.419 \cdot 10^{-5}$	3.949	$4.171 \cdot 10^{-4}$	2.986

Table 1. Errors of the numerical solution of the colliding flow problem for different inf-sup stable finite element approximations, two values for the stabilization parameter, and a sequence of uniformly refined meshes.

The numerical results show the predicted convergence rates. The methods are not very sensitive to the choice of α; in particular, the number of iterations hardly increased when increasing α by a factor of four. Note, that unlike discontinuous Galerkin methods, the jump terms involving α only appear at the interface. For a theory guided choice of α, see [18, 23].

5.2. **Backward facing step.** As a second test case, we consider the backward facing step problem on the geometry depicted in Figure 2. For the hybrid mortar method, the domain is partitioned into three subdomains $\Omega_1 = (-2,-0.5) \times (0.5,1)$, $\Omega_3 = (1,10) \times (0,1)$ and $\Omega_2 = \Omega \backslash (\Omega_1 \cup \Omega_2)$. At the inflow and outflow boundaries, we impose parabolic velocity profiles

$$\boldsymbol{u}(-2,y) = (\,8\,(1-y)\,(y-0.5)\,,0\,) \quad \text{and} \quad \boldsymbol{u}(10,y) = (\,y(1-y),0\,)\,.$$

On the rest of the boundary we apply a no-slip condition. The boundary velocities are chosen such that the compatibility condition $\int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{n}\,dx = 0$ is satisfied, which is due to the incompressibility of the fluid.

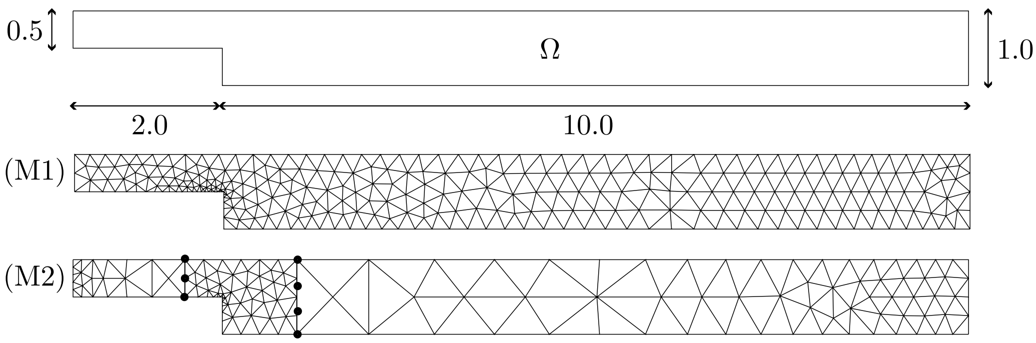

FIGURE 2. Geometry and initial triangulations of the backward facing step domain. Bullets denote interface nodes. (M1) consists of 602 triangular elements and (M2) has 307 triangular elements and 5 interface elements.

For discretization of the backward-facing-step problem, we use the second order Taylor-Hood element. The solutions obtained with third order Taylor-Hood element are used for estimating the discretization error. Again, we run a series of computations on uniformly refined meshes.

In our tests, we compare the hybrid mortar methods on non-matching grids to the corresponding standard methods on a conforming mesh. The two initial meshes (M1) and (M2) used for our computations are shown in Figure 2. They were chosen such that the errors of the two finite element methods are of comparable size on the coarsest level. The initial meshes were also slightly refined towards the inward corner, where a decrease of regularity in the solution can be expected.

The numerically observed convergence rates obtained with the mortar finite element solution for meshes based on (M1) and for a standard conforming finite element method on meshes generated from (M2) are listed in Table 5.2.

The results obtained with the both methods are very similar, i.e., the difference in discretization errors was less than 5% in all computations. As expected, the convergence rates are limited by the corner singularity of the solution.

6. Conclusions

In this article, we proposed and analyzed a hybrid mortar method for Stokes interface problems on non-matching grids. Discrete stability could be shown for a large class of inf-sup

	Standard FEM (M1)				Hybrid Mortar FEM (M2)			
h	L^2 error	rate	energy error	rate	L^2 error	rate	energy error	rate
1.0	$1.380 \cdot 10^{-1}$	–	$2.545 \cdot 10^{-1}$	–	$1.385 \cdot 10^{-1}$	–	$2.688 \cdot 10^{-1}$	–
0.5	$4.288 \cdot 10^{-2}$	1.686	$1.585 \cdot 10^{-1}$	0.683	$3.926 \cdot 10^{-2}$	1.818	$1.543 \cdot 10^{-1}$	0.800
0.25	$1.467 \cdot 10^{-2}$	1.546	$1.075 \cdot 10^{-1}$	0.559	$1.337 \cdot 10^{-2}$	1.554	$1.022 \cdot 10^{-1}$	0.594
0.125	$5.122 \cdot 10^{-3}$	1.518	$7.457 \cdot 10^{-2}$	0.528	$4.822 \cdot 10^{-3}$	1.471	$7.172 \cdot 10^{-2}$	0.510

Table 2. Comparison of numerical errors for the backward facing step problem discretized with second order Taylor-Hood elements. The results on the left are obtained with the standard finite element method on conforming meshes, generated by uniform refinement from (M1). For meshes based on (M2), the hybrid mortar method with stabilization parameter $\alpha = 20$ is used.

stable finite elements, and sufficiently rich spaces of hybrid variables. Optimal energy- and L^2-norm error estimates could be derived, and the theoretical results were illustrated by numerical experiments.

The conditions needed for our stability analysis are relatively weak, i.e., in principle the interface space is required to have at least one degree of freedom for every subdomain interface. This condition is automatically satisfied, if the interface mesh resolves the partition of the skeleton, or when the interface mesh is sufficiently fine. In contrast to other approaches, the subdomain meshes can be chosen completely independent from each other.

Our analysis also covers, as a limiting case, a class of discontinuous Galerkin methods, which can be interpreted as a very fine domain partition.

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